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Operators Commuting with Translation by One
Part III
Perturbation Results for Periodic Differential Operators*

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1. Introduction and Summary

Part I of this paper [1] develops a representation theory for operators on $L_2(-\infty, \infty)$ which commute with translation by one in terms of $L_2(0, 1)$ operator valued functions of θ, $\theta \in [0, 2\pi)$.

Part II [2] lays the ground work for the application to differential operators with periodic coefficients with period taken to be one. The results of these two parts permit us to investigate the properties of the nth order differential operator τ on the infinite axis where it is singular in terms of the properties of the set of discrete operators on (0, 1) defined by the boundary conditions $f^j(1) = e^{i\theta} f^j(0)$, $0 \leqslant j \leqslant n-1$, $\theta \in [0, 2\pi)$.

In this part the perturbation techniques developed by Schwartz [3] and Kramer [4] for examining the spectral properties of discrete operators are extended by this method to obtain conditions on the coefficients of τ sufficient to guarantee that τ in $L_2(-\infty, \infty)$ have a uniformly bounded resolution of the identity, i.e., be spectral in the sense of Dunford [5].

The next few paragraphs summarize the results and terminology needed from Parts I and II.

Let S be the operation of translation by one defined by $(Sf)(t) = f(t+1)$. An operator A which commutes with S is determined by its behavior on $L_2(0, 1)$, in particular by the sequence of operators from $L_2(0, 1)$ to $L_2(j, j+1)$, $j = 0, \pm 1, \cdots$, defined by restricting the domain of A to $L_2(0, 1)$ and, for f in $L_2(0, 1)$, looking only at that part of $(Af)(t)$ in the interval

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$(j, j+1)$. It is more convenient to shift the operands to $(0, 1)$, i.e., to consider the sequence of operators $A(j)$ from $L_2(0, 1)$ to $L_2(0, 1)$ defined by

$$A(j) = S^j \chi_{(j,j+1)} Af, \qquad f \in L_2(0, 1)$$

where $\chi_E(t)$ is the characteristic function of the set E defined by $\chi_E(t) = 1$ if $t \in E$, $\chi_E(t) = 0$ if $t \notin E$. We define the operator function of θ,

$$A'(\theta) = \sum_{k=-\infty}^{\infty} e^{-i\theta k} A(k), \qquad 0 \leqslant \theta \leqslant 2\pi$$

where the sense in which convergence is to be taken is made precise in Part I. The function $A'(\theta)$ is weakly ($=$ strongly) measurable and

$$| A | = \operatorname{ess}_\theta \sup | A'(\theta) |.$$

If $A'(\theta)$ is related to A in the above fashion the notation $A \sim A'(\theta)$ is employed. Conversely, if $A'(\theta)$ is a weakly measurable, $L_2(0, 1)$ operator valued function, $0 \leqslant \theta \leqslant 2\pi$, such that $\operatorname{ess}_\theta \sup | A'(\theta) | < \infty$ then there is a unique bounded operator A such that $AS = SA$ and $A \sim A'(\theta)$.

The algebraic properties of the relation $A \sim A'(\theta)$ are important. If B is another bounded operator such that $BS = SB$ and if $B'(\theta)$ is such that $B \sim B'(\theta)$ then $AB \sim A'(\theta) B'(\theta)$, the pointwise product of the functions $A'(\theta)$ and $B'(\theta)$. Since also $A^* \sim A'(\theta)^*$, it follows readily that A is self-adjoint, normal, a projection, 0, or the identity if and only if the same is true of $A'(\theta)$ for almost all θ, and that A and B commute if and only if $A'(\theta)$ and $B'(\theta)$ commute for almost all θ.

The following theorem from Part I is the main result of this nature used in this paper. In it the space $L_\infty((0, 2\pi); B(L_2(0, 1)))$ is the space of operator valued functions $T(\theta)$ such that for almost all θ in $(0, 2\pi)$, $T(\theta)$ is an operator in $L_2(0, 1)$ which is measurable and such that $| T |_\infty = \operatorname{ess}_\theta \sup | T(\theta) | < \infty$.

THEOREM 5.21 (Part I). *Let A be a bounded linear operator in $L_2(-\infty, \infty)$ such that $AS = SA$ and let $A \sim A'(\theta)$. Then A is a spectral operator (in the sense of Dunford [5]) if and only if $A'(\theta)$ is a spectral operator for a.a. θ with resolution of the identity $E'(\theta, \delta)$, scalar part $B'(\theta)$, and radical part $N'(\theta)$ such that $E'(\theta, \delta)$, $B'(\theta)$, and $N'(\theta) \in L_\infty((0, 2\pi); B(L_2(0, 1)))$, $\sup_{\delta \in \beta} | E'(\theta, \delta) |_\infty < \infty$, and $\lim_{n \to \infty} (|N'(\theta)^n |_\infty)^{1/n} = 0$. If this is the case and $A = B + N$ where B is the scalar part of A and N is the radical part, and A has the resolution of the identity $E(\delta)$, then $B \sim B'(\theta)$, $N \sim N'(\theta)$, and $E(\delta) \sim E'(\theta, \delta)$.*

Let

$$\tau = a_0(t) \left(\frac{d}{dt}\right)^n + a_1(t) \left(\frac{d}{dt}\right)^{n-1} + \cdots + a_n(t)$$

be a formal differential operator with coefficients of period 1, i.e., $a_i(t) = a_i(t+1)$, $0 \leqslant i \leqslant n$. Assume that $a_0(t) \neq 0$, $t \in (-\infty, \infty)$, and that all the coefficients are infinitely differentiable.

In the definitions to follow A^{n-1} is the set of functions with $n-1$ continuous derivatives and with absolutely continuous $(n-1)$st derivative, i.e., all f such that $f^{(n)}(t)$ is defined almost everywhere and integrable over every compact subset of $(-\infty, \infty)$.

Let $T(\tau)$ be the linear operator in $L_2(-\infty, \infty)$ defined by

$$D(T(\tau)) = \{f \text{ in } L_2(-\infty, \infty) \mid f \text{ is in } A^{n-1}, \tau f \text{ is in } L_2(-\infty, \infty)\}$$
$$T(\tau) f = \tau f, f \quad \text{in} \quad D(T(\tau)).$$

The notation T for $T(\tau)$ will also be used when more convenient.

It was shown in Part II that $T(\tau)$ is equal to the closure of its restriction to functions with n derivatives vanishing outside compact sets, i.e., the maximal and minimal operators are equal and we may speak of $T(\tau)$ as *the* operator in $L_2(-\infty, \infty)$ associated with τ.

Associated with τ there is a matrix differential equation $\dot{X} = A(\tau - \lambda)X$ such that $f(t)$ is a solution to $(\tau - \lambda)f = 0$ if and only if the column vector $\mathbf{f}$ formed from f and its first $n-1$ derivatives satisfies the associated matrix differential equation. Any nonsingular solution to the matrix equation can be written in the form $X(t, \lambda) = P(t, \lambda)\, e^{tR(\lambda)}$ where P has period one in t and R is a matrix independent of t. The eigenvalues of $R(\lambda)$ are independent of the particular solution matrix X chosen and are called the *characteristic values* of $\tau - \lambda$. The spectrum of $T(\tau)$, denoted by $\sigma(\tau)$, is just those λ for which $\tau - \lambda$ has a characteristic value of modulus one.

A standing hypothesis is made in Part II that for at least one λ the characteristic values of $\tau - \lambda$ are distinct and that none of them is identically a constant of modulus one. That this hypothesis is satisfied by all operators under consideration in this part can be readily derived from the asymptotic estimates found in McGarvey [6]. Because $e^{R(\lambda)}$ is an analytic function of λ, $\sigma(\tau)$ is thus a collection of arcs analytic except at isolated points.

Since $\sigma(\tau)$ is not the whole plane there is a λ such that

$$R(\lambda, T(\tau)) = (\lambda I - T(\tau))^{-1}$$

is an everywhere defined and bounded operator. The periodicity of τ means that $R(\lambda, T(\tau))\, S = SR(\lambda, T(\tau))$. Hence the theory of Part I can be applied to $R(\lambda, T(\tau))$.

Let $S(\theta, \tau)$ be the operator in $L_2(0, 1)$ with domain $D(S(\theta, \tau)) = \{f \mid f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ exist, all $t \in [0, 1]$, $f^{(n-1)}$ is absolutely continuous on $[0, 1]$, f, $\tau f \in L_2(0, 1)$, $f^{(j)}(1) = e^{i\theta} f^{(j)}(0)$, $0 \leqslant \theta \leqslant n-1\}$ and for $f \in D(S(\theta, \tau))$ let $S(\theta, \tau) f = \tau f$. The notation $S(\theta)$ will also be used for $S(\theta, \tau)$ when more convenient.

The spectrum of $S(\theta, \tau)$ is just those λ for which $e^{i\theta}$ is a characteristic value to $\tau - \lambda$ and $R(\lambda, S(\theta, \tau))$ is compact for every $\lambda \in \rho(S(\theta, \tau))$. That $T(\tau)$ might be closely related to $S(\theta), \tau)$ for $0 \leqslant \theta \leqslant 2\pi$ is suggested by the fact that $\sigma(\tau) = \bigcup_{0 \leqslant \theta \leqslant 2\pi} \sigma(S(\theta, \tau))$. There is indeed a close relationship, for

$$R(\lambda, T(\tau)) \sim R(\lambda, S(\theta, \tau)), \qquad \lambda \notin \sigma(\tau).$$

Furthermore it is shown in Part II that for almost all θ, $S(\theta, \tau)$ can not have a nilponent part which greatly simplifies the application of the theory of Part I.

Let R be the ring consisting of all sets which are the finite union of half closed rectangles. Let $e(\theta, \delta)$ be the spectral projection algebra for $S(\theta, \tau)$ defined by contour integration of the resolvent of $S(\theta, \tau)$. We have the following basic result from Part II.

THEOREM 3.5, (Part II). *Under the standing hypothesis concerning the characteristic values, $T = T(\tau)$ is a spectral operator if and only if*

$$\sup_{\delta \in R} \operatorname*{ess\,sup}_{0 \leqslant \theta < 2\pi} | e(\theta, \delta) | < \infty$$

and $e(\theta, Z) = I$, a.a. θ. If it is spectral it is of scalar type.

We shall occasionally have to refer to other results from Part II but they are not essential for a first reading.

This result brings the spectral theory of periodic differential operators in $L_2(-\infty, \infty)$ within the reach of the work of Schwartz [3] and Kramer [4] who have developed perturbation techniques which show that regular differential operators with leading coefficient 1 subject to a large class of boundary conditions are spectral operators. The application of their techniques is developed in Sections 3 and 4. The discussion is limited to the case $a_0(t) = 1$ or equivalently (see Theorem 2.1 below) $\arg a_0(t) = \text{constant}$.

The resulting theorems for τ in $L_2(-\infty, \infty)$ are weaker than Schwartz and Kramer's result for the reasons discussed in the next few paragraphs.

It is essential in their proofs that the unperturbed operator, T, have a discrete spectrum and that $\dim E(\lambda, T) < \infty$ for every λ in $\sigma(T)$ where $E(\cdot, T)$ is the resolution of the identity of T and $\dim E(\lambda, T)$ is the dimension of the range of $E(\lambda, T)$. If $\lambda \in \sigma(T)$ is such that $\dim E(\lambda, T) = m$, m finite, we consider λ to be m distinct coincident points in $\sigma(T)$. An important quantity in Schwartz and Kramer's work is $d(\lambda)$ defined for $\lambda \in \sigma(T)$ as the distance from λ to the rest of $\sigma(T)$. If λ in $\sigma(T)$ coincides with another point in $\sigma(T)$ then $d(\lambda) = 0$.

Speaking roughly, when T is perturbed to $T + B$ each point λ in $\sigma(T)$ moves into a point $\lambda(B)$ in $\sigma(T + B)$. The quantity $d(\lambda)$ is important in estimating first $| E(\lambda(B), T + B) - E(\lambda, T) |$, and then, by the triangle inequality,

$|E(\lambda(B), T+B)|$. The smaller $d(\lambda)$ the larger $|E(\lambda(B), T+B) - E(\lambda, T)|$ may be and if $d(\lambda) = 0$ no estimate can be made except that

$$|E(\lambda(B), T+B)| < \infty.$$

Since Schwartz and Kramer are only interested in proving that $|E(\delta, T+B)|$ is uniformly bounded in δ they are able to consider only the asymptotic behavior of $d(\lambda)$ and ignore any finite number of λ in $\sigma(T)$. Thus they can allow $\dim E(\lambda, T) = m$, $1 < m < \infty$ for up to a finite number of λ in $\sigma(T)$. This in turn permits them to work with perturbations which are "small" in a very weak sense of this word relative to the unperturbed operator.

However, for our purposes it is necessary to perturb the *class* of operators $S(\theta, \tau)$, $0 \leqslant \theta < 2\pi$, to $S(\theta, \tau + \tau_1)$ and find an upper bound on $|E(\delta, S(\theta, \tau + \tau_1))|$ which is uniform in θ. This makes it necessary, among other things, to require that

$$\min_{0 \leqslant \theta < 2\pi} \min_{\lambda \in \sigma(S(\theta,\tau))} d(\lambda) > 0.$$

In particular this makes it impossible for us at present to perturb $(id/dt)^n$ for n even although it is possible to perturb $(id/dt)^n + ic(id/dt)^{n-1}$ for c real, $c \neq 0$, and to perturb $(id/dt)^n$ for n odd.

In compensation, if the unperturbed operator is a polynomial in (d/dt), i.e., $\tau = P(d/dt)$ where $P(x)$ is a polynomial, then $S(\theta, \tau) = P(S(\theta, d/dt))$ which means that $S(\theta, \tau)$ is normal, $0 \leqslant \theta < 2\pi$, and that

$$\sigma(S(\theta, \tau)) = P(\sigma(S(\theta, d/dt))) = \{P(-i(\theta + 2\pi j))\}, \qquad j = 0, \pm 1, \cdots,$$

and thus we are spared the laborious analysis of the spectrum of the unperturbed operator that the study of general boundary conditions imposes on Schwartz and Kramer.

The following theorem is a sample of the type of perturbation results that are proved.

THEOREM 1.1. *Let τ be a periodic differential operator of the form*

$$\tau = \left(i\frac{d}{dt}\right)^2 + ic\left(i\frac{d}{dt}\right) + b_1(t)\left(i\frac{d}{dt}\right) + b_2(t)$$

where c is a real constant. If there exists a constant d such that

$$\sup_t |b_1(t)| \cdot \frac{|c + 2\pi i|}{4\pi |c|} + \sup_t |b_2(t) - d| \cdot \frac{1}{2\pi |c|} < \tfrac{1}{2}$$

then $T(\tau)$ is a spectral operator of scalar type.

Results of a similar nature are stated for τ of arbitrary order. Some further results are obtained by considering the dependence on the real constant a of the operator

$$-\left(\frac{d}{dt}\right)^2 + a\left(\frac{d}{dt}\right) + q(t)$$

where $q(t)$ has period 1. The relationship between the number of bounded, connected components of the spectrum of $-(d/dt)^2 + q(t)$ and the number of values of a for which $-(d/dt)^2 + a(d/dt) + q(t)$ is not spectral is investigated. These are summarized in Theorems 4.14 and 4.15.

2. Changes of Variable

It is useful at times below to make changes of variable which reduce τ to a standard form. Our discussion is patterned after that in some unpublished work of Dunford and Schwartz. It applies as well to $L_p(-\infty, \infty)$, $1 \leqslant p \leqslant \infty$ although our direct concern is only with $L_2(-\infty, \infty)$. First multiply τ by a suitable scalar to bring it into the form

$$\tau = (ir(t))^n e^{i\theta(t)} \left(\frac{d}{dt}\right)^n + a_1(t)\left(\frac{d}{dt}\right)^{n-1} + \cdots + a_n(t)$$

where $\theta(t)$ is real, $r(t) > 0$, and $\int_0^1 r(t)^{-1}\, dt = 1$. Let $g(s) = \int_0^s r(u)^{-1}\, du$. Since $r(u) > 0$, $G(s)$ is monotone increasing and, since $r(u)$ has period 1, $g(s) - s$ has period 1 and $\lim_{s\to\pm\infty} g(s) = \pm\infty$. For $-\infty < t < \infty$ let $s = h(t)$ be the unique solution of $t = g(s)$, i.e., $h(t) = \operatorname{arc} g(t)$.

It is not difficult to show that the change of independent variable $(Uf)(t) = f(h(t))$ defines a bounded 1-1 transformation of $L_p(-\infty, \infty)$ onto itself and has the bounded inverse $(U^{-1}f)(s) = f(g(s))$. It follows that $T(\tau)$ has a bounded inverse or is a spectral operator if and only if $UT(\tau)\, U^{-1}$ has a bounded inverse or is a spectral operator. Computation shows that

$$(UT(\tau)\, U^{-1}f)(t) = \tau_1 f(t)$$

where

$$\begin{aligned}\tau_1 &= i^n e^{i\theta}(h(t))\left(\frac{d}{dt}\right)^n \\ &\quad + \left[a_1(h(t))\, r(h(t))^{-(n-1)} - \frac{n(n-1)}{2} i n e^{i\theta(h(t))}\, r'(h(t))\right]\left(\frac{d}{dt}\right)^{n-1} \\ &\quad + \cdots + b_n(t).\end{aligned}$$

If $a(s)$ has period 1 then $a(h(t+1)) = a(h(t)+1) = a(h(t))$ so that $a(h(t))$ will also have period 1. Thus all the coefficients of τ_1 have period 1.

Consider next $VT(\tau_1)\,V^{-1}$ where V is a change of *dependent* variable of the form $(Vf)(t) = e^{b(t)}f(t)$ with inverse $(V^{-1}f)(t) = e^{-b(t)}f(t)$ where $b(t)$ will be picked shortly. *The discussion will be restricted to the case* $\theta(t) \equiv 0$ although it is interesting to carry through the argument for general $\theta(t)$. It is worth mentioning that a curious pathology results if $\int_0^1 e^{-i\theta(h(t))}\,dt = 0$; in this case it is possible to reduce the second coefficient to a constant only under very special circumstances. Notice that for V to be an automorphism of $L_p(-\infty, \infty)$ into itself it is necessary that $\operatorname{Re}(b(t))$ be uniformly bounded above and below. Also if $b(t)$ is sufficiently differentiable $f \in A^{n-1}$ if and only if $Vf \in A^{n-1}$. Assuming $b(t)$ is sufficiently differentiable, calculation shows that

$$VT(\tau_1)\,V^{-1}f = \tau_2 f$$

where

$$\begin{aligned}
\tau_1 &= i^n \left[\frac{d}{dt} - b'(t) \right]^n \\
&\quad + \left[a_1(h(t))\,r(h(t))^{-(n-1)} - \frac{n(n-1)}{2}\, i^n r'(h(t))\right] \left[\frac{d}{dt} - b'(t)\right]^{n-1} \\
&\quad + \cdots + b_n(t) \\
&= i^n \left(\frac{d}{dt}\right)^n + \Big[a_1(h(t))\,r(h(t))^{-(n-1)} \\
&\qquad\qquad - \frac{n(n-1)}{2}\, i^n r'(h(t)) - n i^n b'(t)\Big] \left(\frac{d}{dt}\right)^{n-1} \\
&\quad + \cdots + c_n(t).
\end{aligned}$$

We wish τ_2 to also have coefficients of period 1 and the first expression for τ_2 displayed above shows that this will be true if and only if $b'(t)$ has period 1. Thus $b(t) = b_1(t) + at$ where $b_1(t)$ is periodic and $a = \int_0^1 b'(t)\,dt$. In order for $\operatorname{Re} b(t)$ to be bounded above and below it is necessary and sufficient that $\operatorname{Re}(a) = 0$. Thus the choice of $b'(t)$ is subject to the constraints that $b'(t)$ be periodic with period 1 and $\operatorname{Re} \int_0^1 b'(t)\,dt = 0$. Let us try to make the second coefficient a constant $i^n c$. Then

$$b'(t) = n^{-1} i^{-n} a_1(h(t))\,r(h(t))^{-(n-1)} - \frac{(n-1)}{2}\, r'(h(t)) - n^{-1}c$$

and the constraint $\operatorname{Re} \int_0^1 b'(t)\,dt = 0$ requires that

$$\operatorname{Re}(c) = \operatorname{Re} \int_0^1 i^{-n} a_1(h(t))\,r(h(t))^{-(n-1)}\,dt - \frac{n(n-1)}{2} \int_0^1 r'(h(t))\,dt.$$

Now

$$\int_0^1 r'(h)\,t))\,dt = \int_0^1 r'(s) \left(\frac{dt}{ds}\right) ds$$

where $s = h(t)$. The definition of $h(t)$ yields $(dt/ds) = r(s)^{-1}$. Hence

$$\int_0^1 r'(h(t))\, dt = \int_0^1 r'(s)\, r(s)^{-1}\, ds = \log r(1) - \log r(0) = 0$$

since $r(t)$ is positive and periodic. The change of variables $t \to s = h(t)$ in the first integral displayed above yields

$$\begin{aligned}\int_0^1 i^{-n} a_1(h(t))\, r(h,\ (t))^{-(n-1}\, dt &= \int_0^1 i^{-n} a_1(s)\, r(s)^{-n}\, ds \\ &= \int_0^1 \frac{a_1(s)}{a_0(s)}\, ds\end{aligned}$$

where $a_0(s) = (ir(t))^n$ is the leading coefficient of τ. The constraint on c thus becomes

$$\mathrm{Re}\,(c) = \mathrm{Re} \int_C^1 \frac{a_1(s)}{a_0(s)}\, ds$$

and the imaginary part of c can be arbitrary.

This discussion of changes of variable is summarized in the following theorem.

THEOREM 2.1. *Let*

$$\tau = a_0(t) \left(\frac{d}{dt}\right)^n + a_1(t) \left(\frac{d}{dt}\right)^{n-1} + \cdots + a_n(t)$$

where $a_0(t) = (ir(t))^n e^{i\theta(t)}$, $r(t) > 0$, $\theta(t)$ *real,* $\int_0^1 r(t)^{-1}\, dt = 1$. *Then there is an automorphism* U *of* $L_2(-\infty, \infty)$ *onto itself such that* $UT(\tau)\, U^{-1} = T(\tau_1)$ *where* τ_1 *is of the form*

$$\tau_1 = i^n e^{i\theta(h(t))} \left(\frac{d}{dt}\right)^n + b_1(t) \left(\frac{d}{dt}\right)^{n-1} + \cdots + b_n(t)$$

and has coefficients of period 1. Let $\theta(t) \equiv 0$. *Then there is an automorphism* $W(= VU)$ *of* $L_2(-\infty, \infty)$ *onto itself, such that* $WT(\tau)\, W^{-1} = T(\tau_2)$ *where*

$$\tau_2 = i^n \left(\frac{d}{dt}\right)^n + i^n c \left(\frac{d}{dt}\right)^{n-1} + c_2(t) \left(\frac{d}{dt}\right)^{n-2} + \cdots + c_n(t);$$

all the coefficients of τ_2 *have period 1;*

$$\mathrm{Re}\,(c) = \mathrm{Re} \int_0^1 a_1(s)\, a_0(s)^{-1}\, ds;$$

and $\mathrm{Im}\,(c)$ *is arbitrary.*

3. Perturbation Results, nth Order Case

Schwartz [3] and Kramer [4] have established the following result:

THEOREM 3.1 (Schwartz, Kramer). *Let T be a discrete spectral operator in a weakly complete B space X. Let E be the resolution of the identity of T. Suppose that for $\lambda \in \sigma(T)$ the projection $E(\lambda)$ has 1-dimensional range for all but a finite number of λ. Let $\lambda_0 \in \rho(T)$; let $0 \leqslant v < 1$, and let P be an operator such that $D(P) \supseteq D((T - \lambda_0 I)^v)$, and such that $P(T - \lambda_0 I)^{-v}$ is bounded. Let $\{\lambda_n\}$ be an enumeration of $\sigma(T)$ and let d_n be the distance from λ_n to $\sigma(T) - \{\lambda_n\}$.*
Then, if

$$\sum_{n=1}^{\infty} d_n^{-1}(|\lambda_n| + d_n)^v < \infty \tag{1}$$

the operator $T + P$ is a discrete spectral operator. The spectrum of $T + P$ may be enumerated as $\sigma(T + P) = \{\mu_n\}$ where $|\lambda_n - \mu_n| < d_n/2$ for all but a finite number of n. If X is Hilbert space the condition that (1) hold may be replaced by the condition

$$\sum_{n=1}^{\infty} d_n^{-2}(|\lambda_n| + d_n)^{2v} < \infty. \tag{2}$$

It is convenient to introduce some terminology. By employing the convention that a set U is an open neighborhood of ∞ if and only if U' is a compact set, the definitions below extend to the point $\lambda = \infty$.

DEFINITION 3.1. Let Θ be a Borel subset of $[0, 2\pi)$. For a point λ we will say that $e(\theta, \cdot)$ is uniformly essentially bounded near λ for θ in Θ if there is an open neighborhood Δ of λ such that

$$\sup_{\delta \in R(\Delta)} \operatorname*{ess\,sup}_{\theta \in \Theta} | e(\theta, \delta) | < \infty.$$

We will say that $e(\theta, \cdot)$ is uniformly essentially bounded on compact sets for θ in Θ if $e(\theta, \cdot)$ is uniformly essentially bounded for $\theta \in \Theta$ near every finite λ. If $\Theta = [0, 2\pi)$ we will often say merely $e(\theta, \cdot)$ is uniformly essentially bounded near λ and $e(\theta, \cdot)$ is uniformly essentially bounded on compact sets.

Our condition defining uniform essential boundedness on compact sets for $\theta \in \Theta$ is equivalent to the condition that

$$\sup_{\delta \in R(\Delta)} \operatorname*{ess\,sup}_{\theta \in \Theta} | e(\theta, \delta) | < \infty$$

for every compact set Δ. It is also true that $e(\theta, \cdot)$ is both uniformly essentially bounded on compact sets for θ in Θ and uniformly essentially bounded near ∞ for θ in Θ if and only if

$$\sup_{\delta \in \beta} \operatorname{ess\,sup}_{\theta \in \Theta} | e(\theta, \delta) | < \infty.$$

DEFINITION 3.2. If $e(\theta, \cdot)$ is uniformly essentially bounded near ∞ and $e(\theta, Z) = I$ for Borel almost all θ, $0 \leqslant \theta < 2\pi$ then T will be called *spectral at* ∞. If $e(\theta, \cdot)$ is uniformly bounded on compact sets then T will be called *spectral on compact sets.*

Schwartz and Kramer's results must be modified somewhat to suit our purposes since we must find bounds on $| e(\theta, \delta) |$ uniform in θ, $0 \leqslant \theta < 2\pi$. In order to get results allowing as large a perturbation as possible we have taken a bit of care to estimate various bounds appearing in the proofs below. Another problem lies in the fact that for n even and for $\theta = 0$ or π the operator $S(\theta, (d/dt)^n)$ has an infinite number of points in the spectrum of multiplicity 2. Hence to deal with nonnormal even order operators at all we must perturb something besides $(d/dt)^n$.

In compensation the nature of the boundary conditions we are interested in is such that we are spared the lengthy argumentation needed to apply Schwartz and Kramer's abstract perturbation theorem to differential operators on [0, 1] with arbitrary boundary conditions.

Most of the theorems below which prove that a certain class of $T(\tau)$ are spectral do so by perturbing operators with constant coefficients. The conditions of Theorem 3.6 below are stated with this in mind. We have also included Theorems 3.4 and 3.5 in order to perturb operators which are spectral but not with constant coefficients.

The fact that for $\lambda \in \rho(T)$, $R(\lambda, S(\theta))$ is an analytic function of θ reduces greatly the amount of pathology that can appear.

Because of this analyticity to show that T is spectral on compact sets it is sufficient to show that $\dim e(\theta, \lambda) = 1$ for every $\lambda \in \sigma(\theta)$ and every $\theta \in [0, 2\pi)$.

A necessary condition is that $S(\theta)$ be scalar, i.e., that there be no f such that $(S(\theta) - \lambda) f \neq 0$ and $(S(\theta) - \lambda)^k f = 0$ for some k. Such an f is called a *proper generalized eigenvector*. ("Scalar" in our terminology does not necessarily mean spectral—see Part II.)

Thus the only case that is unresolved is that where for some λ, $\dim e(\theta, \lambda) > 1$, and there are no proper generalized eigenvectors. In Section 4, Theorem 4.4 it will be shown that for an interesting class of second order operators this latter condition can not occur and hence

$$\dim e(\theta, \lambda) L_2(0, 1) \leqslant 1$$

for all real θ and all λ will be a necessary and sufficient condition that T be spectral.

The sufficiency theorems that operators be spectral on compact sets stated below (Theorem 3.5, Theorem 3.6a) are basically proofs that

$$\dim e(\theta, \lambda) L_2(0, 1) \leqslant 1.$$

These proofs are based on Lemma 3.1 below.

The treatment of the condition of being spectral at infinity is closer to that of Schwartz and Kramer, being essentially asymptotic in nature. The main difference is one of emphasis since we have chosen to look mostly at perturbations of differential operators with constant coefficients. The spectra of such operators subject to the boundary conditions $\mathbf{f}(1) = e^{i\theta}\mathbf{f}(0)$ are very easily characterized since for these boundary conditions they are polynomials in the operator $i(d/dt)$ subject to the same boundary conditions.

As a reading of Theorem 3.6(b) will reveal, the perturbation restrictions that are made in order to insure that the perturbed operator be spectral at infinity are on the *order* of the perturbation term.

The application of Theorem 3.6(b) becomes a study in the asymptotic behavior of polynomials with complex coefficients for values of the argument which are real and large in absolute value. Some results along this line are proved in the final parts of the section and are summarized in Theorem 3.7.

LEMMA 3.1. *Let $\lambda_0 \in \sigma(S(\theta_0))$. A sufficient condition that there exist a neighborhood V of θ_0 such that $e(\theta, \cdot)$, $\theta \in V$, is uniformly essentially bounded near* λ_0 is that

$$\dim e(\theta_0, \lambda_0) L_2(0, 1) = 1.$$

A necessary condition is that $S(\theta_0) \mid e(\theta_0, \lambda_0) L_2(0, 1)$[1] *be scalar*

PROOF. If $\dim e(\theta_0, \lambda_0) L_2(0, 1) = 1$ it follows from Theorem 2.7 of Part II that there is a neighborhood V of θ_0 and a neighborhood U of λ_0 such that $e(\theta, U)$ is an analytic function of θ, $\theta \in V$. Since $\lim_{\theta \to \theta_0} e(\theta, U) = e(\theta_0, U) = e(\theta_0, \lambda_0)$ it follows that $\dim e(\theta, U) = 1$, $\theta \in V$. Thus for $\delta \subseteq V$, $e(\theta, \delta) = 0$ or $e(\theta, \delta) = e(\theta, V)$ and hence

$$\sup_{\substack{\delta \subseteq U \\ \theta \in V}} | e(\theta, \delta) | = \sup_{\theta \in V} | e(\theta, U) | < \infty.$$

This proves the first assertion of the lemma.

To prove the second assertion we can assume with no loss of generality that $\lambda_0 = 0$, i.e., $S(\theta_0) \mid e(\theta_0, 0) L_2(0, 1)$ is not of scalar type. Thus

$$m = \dim e(\theta_0, 0) L_2(0, 1) > 1.$$

[1] The restriction of $S(\theta_0)$ to the subspace $e(\theta_0, \lambda_0) L_2(0, 1)$.

It follows from Lemma 3.3 of Part II that there is a neighborhood V of θ_0 and a neighborhood U of 0 such that for $\theta \in V - \{\theta_0\}$,

$$\sigma(S(\theta)) \cap U = \{\lambda_1(\theta), \cdots, \lambda_m(\theta)\},$$

where

$$\lambda_i(\theta) \neq \lambda_j(\theta), \qquad i \neq j, \qquad \theta \neq \theta_0, \qquad \lim_{\theta \to \theta_0} \lambda_i(\theta) = 0,$$

and

$$\dim e(\theta, \lambda_i(\theta)) L_2(0, 1) = 1.$$

Thus

$$S(\theta)e(\theta, U) = \sum_{i=1}^{m} \lambda_i (\theta) e(\theta, \lambda_i(\theta))$$

The assumption that $S(\theta_0) \mid e(\theta_0, 0) L_2(0, 1)$ is not of scalar type implies the existence of an element $x \in L_2(0, 1)$ such that $S(\theta_0)\, e(\theta_0, 0)\, x \neq 0$. But

$$S(\theta_0)e(\theta_0, 0)x = \lim_{\theta \to \theta_0} S(\theta)e(\theta, U)x = \lim_{\theta \to \theta_0} \sum_{i=1}^{m} \lambda_i(\theta)e(\theta, \lambda_i(\theta))x.$$

Since $\lim_{\theta \to \theta_0} \lambda_i(\theta) = 0$, this implies that

$$\lim_{\theta \to \theta_0} \max_{1 \leqslant i \leqslant m} | e(\theta, \lambda_i(\theta)) | = \infty$$

which implies the desired result. Q.E.D.

The following theorem is a corollary to the results above.

THEOREM 3.2. *If* $\dim e(\theta, \lambda) L_2(0, 1) = 1$, *for all* $\lambda \in \sigma(S(\theta))$ *and all* θ, $\cdot\, 0 \leqslant \theta < 2\pi$, *then T is spectral on compact sets. If for some θ, $0 \leqslant \theta < 2\pi$, $S(\theta)$ is not scalar, then T is not spectral on compact sets.*

Of course if T is not spectral on compact sets it is not spectral.

All the perturbation results to follow will be applications of the following lemma.

LEMMA 3.2. *Let P and T be linear operators in a B-space X such that $D(P) \supseteq D(T)$. Let $\mu \in \rho(T)$ be such that $| PR(\mu, T) | < 1$. Then $1 \in \rho(PR(\mu, T))$, $\mu \in \rho(T + P)$ and $R(\mu, T + P) = R(\mu, T)\,(I - PR(\mu, T))^{-1}$. If T is discrete then so is P. If C is an admissible contour in $\rho(T)$ and C^0 the open set enclosed by C, if $\sup_{\mu \in C} | PR(\mu, T) | < 1$, then $C \subseteq \rho(T + P)$ and if one of* $\dim E(C^0, T)\, X$, $\dim E(C^0, T + P)\, X$ *is finite then*

$$\dim E(C^0, T)\, X = \dim E(C^0, T + P)\, X.$$

PROOF. Since

$$| PR(\mu, T) | = M < 1, \qquad (I - PR(\mu, T))^{-1} = \sum_{n=0}^{\infty} (PR(\mu, T))^n$$

exists and

$$| (I - PR(\mu, T))^{-1} | \leqslant (1 - M)^{-1}, \qquad \mu \in C.$$

Let

$$B(\mu) = R(\mu, T)(I - PR(\mu, T))^{-1}, \qquad \mu \in C.$$

Then the range of $B(\mu)$ is contained in $D(T)$ which equals $D(T + P)$ so that $(\mu I - T - P)\, B(\mu)\, x$ is well defined for all x in X. We have

$$\begin{aligned}(\mu I - T - P)\, B(\mu) &= (\mu I - T)\, R(\mu, T)(I - PR(\mu, T))^{-1} \\ &\quad - PR(\mu, T)(I - PR(\mu, T))^{-1} \\ &= (I - PR(\mu, T))(I - PR(\mu, T))^{-1} = I.\end{aligned}$$

Furthermore for x in $D(T + P)$,

$$(\mu I - T - P)\, x = (I - PR(\mu, T))(\mu I - T)\, x$$

so

$$\begin{aligned}B(\mu)(\mu I - T - P)\, x &= R(\mu, T)(I - PR(\mu, T))^{-1}(I - PR(\mu, T))(\mu I - T)\, x \\ &= x, \qquad x \in D(T + P),\end{aligned}$$

and hence $B(\mu) = R(\mu, T + P)$. The product of a bounded operator and a compact operator is compact, hence if $R(\mu, T)$ is compact then so is $R(\mu, T + P)$, i.e., if T is discrete then so is $T + P$.

Since C is compact and $R(\mu, T)$ is analytic for $\mu \in \rho(T)$, there exists a constant M' such that

$$\sup_{\mu \in C} | R(\mu, T) | \leqslant M'.$$

Consider now the operators $T + aP$, $0 \leqslant a \leqslant 1$. The existence of

$$R(\mu, T + aP) = R(\mu, T)(I - aPR(\mu, T))^{-1}, \qquad \mu \in C$$

follows as above and

$$\begin{aligned}&R(\mu, T + aP) - R(\mu, T + a'P) \\ &= (a' - a)\, R(\mu, T)\, PR(\mu, T)(I - aPR(\mu, T))^{-1}(I - a'PR(\mu, T))^{-1}, \ \mu \in C,\end{aligned}$$

so that

$$| R(\mu, T + aP) - R(\mu, T + a'P | \leqslant | a' - a |\, M'M(1 - M)^{-2}, \qquad \mu \in C,$$

and hence

$$\begin{aligned}&| E(C^0, T + AP) - E(C^0, T + a'P) | \\ &\qquad \leqslant \frac{1}{2\pi} \left| \int (R(\mu, T + aP) - R(\mu, T + a'P))\, d\mu \right| \\ &\qquad \leqslant \frac{L(C)}{2\pi} | a' - a |\, M'M(1 - M)^{-2}\end{aligned}$$

where $L(C)$ is the length of C. Thus $E(C^0, T + aP)$ is a continuous projection valued function of a, $0 \leqslant a \leqslant 1$. If two projections E_1 and E_2 satisfy

$$| E_1 - E_2 | \leqslant \min (| E_1 |^{-1}, | E_2 |^{-1})$$

then both have the same dimension. Therefore, if $\dim E(C^0, T + aP) X < \infty$ for some a, $0 \leqslant a \leqslant 1$, then

$$\dim E(C^0, T + aP) X = \dim E(C^0, T + a'P) X$$

for all a and a' in the interval $[0, 1]$, in particular, setting $a = 0$, $a' = 1$ yields the desired result. Q.E.D.

Our principle interest in Lemma 3.2 is that it is a means of showing $\dim e(\theta, \lambda, \tau) L_2(0, 1) = 1, \lambda \in \sigma(S(\theta, \tau)), 0 \leqslant \theta < 2\pi$ for suitable τ and hence showing that $T(\tau)$ is spectral on compact sets. An interesting application is the following result which has already been used in Part II.

THEOREM 3.3. *If* $\operatorname{var}_{0 \leqslant t \leqslant 1} \arg a_0(t) < \pi$ *then* $\rho(\tau) \neq \phi$.

PROOF. Let $\operatorname{var}_{0 \leqslant t \leqslant 1} \arg a_0(t) = M < \pi$. Then (Theorem 2.1), we lose no generality in assuming that τ has the form

$$\tau = e^{i\theta t} \left(\frac{id}{dt}\right)^n + a_1(t) \left(\frac{id}{dt}\right)^{n-1} + \cdots + a_n(t)$$

where $\theta(t)$ is real,

$$-\frac{\pi}{2} < -\frac{M}{2} = \min \theta(t) \leqslant \max \theta(t) = \frac{M}{2} < \frac{\pi}{2}.$$

Let $a = (\cos M/2)^{-1}$. Then $\max | a - e^{i\theta(t)} | < a$.

Let $D = T(id/dt)$, the maximal closed linear operator defined by the formal differential operator id/dt in $L_2(-\infty, \infty)$. Then D is self-adjoint. For any function $f \in L_\infty(-\infty, \infty)$ let $[f]$ be the operation in $L_2(-\infty, \infty)$ of multiplication by the function f. Then $| [f] | = | f |_\infty$. Let

$$S = aD^n + [e^{i\theta(t)} - a] D^n + [a_1] D^{n-1} + \cdots + [a_n].$$

Then it is easily seen that $T(\tau) \supseteq S$ so that if we can show that $(S - \lambda I)^{-1}$ exists for some λ then $T(\tau) = S$ (since $T(\tau)$ has no point spectrum) and $\rho(T(\tau)) = \rho(\tau) \neq \phi$. Let $T_0 = aD^n$ and

$$P = [e^{i\theta(t)} - a] D^n + [a_1] D^{n-1} + \cdots + [a_n].$$

Since

$$D(T_0) = \{f \mid D^i f \in L_2(-\infty, \infty), 0 \leqslant i \leqslant n\}, \qquad D(P) = D(T_0).$$

Using the operational calculus for self-adjoint operators, for $\sigma \neq 0$ and real and $0 \leqslant k \leqslant n$, $D^k(aD^n - i\sigma^n)^{-1} = f(D)$ where $f(z) = z^k(az^n - i\sigma^n)^{-1}$, and

$$
\begin{aligned}
| D^k(aD^n - i\sigma^n)^{-1} | &= \sup_{z \text{ real}} | z^k(az^n - i\sigma^n)^{-1} | \\
&= \sup_{z \text{ real}} \left| \sigma^{k-n} \left(\frac{z}{\sigma}\right)^k \left(a\left(\frac{z}{\sigma}\right)^n - i\right)^{-1} \right| \\
&= | \sigma |^{k-n} \sup_{z \text{ real}} | z^k(az^n - i) | = | \sigma |^{k-n} M_k
\end{aligned}
$$

where

$$M_k = \sup_{z \text{ real}} | z^k(az^n - i) | < \infty.$$

It is easy to show $M_n = a^{-1}$, and thus

$$| PR(i\sigma^n, T_0) | \leqslant \frac{| [e^{i\theta(t)} - a] |}{a} + | \sigma |^{-1} | [a_1] | M_{n-1} + \cdots + | \sigma |^{-n} | [a_n] | M_0 .$$

The constant a has been so chosen that $| [e^{i\theta(t)} - a] | < a$ and hence for $| \sigma |$ sufficiently large, $| PR(i\sigma^n, T_0) | < 1$. Thus, using Lemma 3.2, for $| \sigma |$ sufficiently large, $R(i\sigma^n, S)$ exists. Q.E.D.

THEOREM 3.4. *Let T be a discrete spectral operator in a Banach space X with resolution of the identity E. Let $M = \sup_{\delta\in\beta} | E(\delta) |$ where β is the field of Borel sets. Let $\{\lambda_i\}$, $1 \leqslant i < \infty$, be an enumeration of $\sigma(T)$. Suppose* $\dim E(\lambda_i) X = 1$, $1 \leqslant i < \infty$. *Let d_i be the distance from λ_i to $\sigma(T) - \{\lambda_i\}$. Suppose that for some ν, $0 \leqslant \nu < 1$,*

$$\sup_{1 \leqslant i < \infty} \frac{| \lambda_i |^\nu}{d_i} = M_1 < \infty.$$

Let P be an operator such that $D(P) \supseteq D(T^\nu)$ and $PT^{-\nu}$ is bounded. If $| PT^{-\nu} | < (8MM_1)^{-1}$ then $T + P$ is a discrete operator and $\sigma(T + P)$ can be enumerated as $\sigma(T + P) = \mu_i$, $1 \leqslant i < \infty$ where $| \mu_i - \lambda_i | < d_i/2$. The projections $E(\mu_i, T + P)$ are one dimensional. If X is Hilbert space and T normal then the condition $| PT^{-\nu} | < (2M_1)^{-1}$ is sufficient to imply the same conclusion.

PROOF. The relation $D(T^\nu) \subseteq D(T)$ implies $D(P) \subseteq D(T)$. Let C_i be the circle about λ_i with radius $d_i/2$. The theorem will follow from Lemma 3.2 once we show that $| PR(\mu, T) | < 1$ for μ either on one of the C_i or exterior to all of them. Writing $PR(\mu, T) = PT^{-\nu}T^\nu R(\mu, T)$ it is sufficient to show

$|T^\nu R(\mu, T)| < |PT^{-\nu}|^{-1}$. From the operational calculus for scalar type spectral operators (see Theorem 2.24 of Part I, which can be extended to the unbounded case),

$$|T^\nu R(\mu, T)| \leqslant 4M \sup_i \frac{|\lambda_i|^\nu}{|\mu - \lambda_i|}$$

and since $|\mu - \lambda_i| \geqslant d_i/2$,

$$|T^\nu R(\mu, T)| < 8MM_1 < |PT^{-\nu}|^{-1}$$

by hypothesis. If X is Hilbert space and T is normal then E is self adjoint so

$$|T^\nu R(\mu, T)| = \sup_{1 \leqslant i < \infty} \frac{|\lambda_i|^\nu}{|\mu - \lambda_i|}$$

and the proof proceeds as before. Q.E.D.

THEOREM 3.5. *Let τ_1 be a regular differential operator with periodic coefficients such that $T(\tau_1)$ is spectral. Let τ_2 be a (regular or irregular) differential operator with periodic coefficients of degree less than that of τ_1. Suppose that for every θ, $0 \leqslant \theta < 2\pi$, $e(\theta, \lambda, \tau_1)$ is one dimensional, $\lambda \in \sigma(S(\theta, \tau_1))$. For $\lambda \in \sigma(S(\theta, \tau_1))$ let $d(\lambda, \theta)$ be the distance from λ to the rest of $\sigma(S(\theta, \tau_1))$. Let*

$$M' = \sup_{\substack{0 \leqslant \theta < 2\pi \\ \delta \in \beta}} |e(\theta, \delta, \tau_1)|.$$

Suppose that for some ν, $0 \leqslant \nu < 1$,

$$M_1' = \sup_{\substack{0 \leqslant \theta < 2\pi \\ \lambda \in \sigma(S(\theta, \tau_1))}} \frac{|\lambda|^\nu}{d(\lambda, \theta)} < \infty.$$

If

$$\sup_{0 \leqslant \theta < 2\pi} |S(\theta, \tau_2) S(\theta, \tau_1)^{-\nu}| < (8M'M_1')^{-1},$$

then $T(\tau_1 + \tau_2)$ is spectral on compact sets. If τ_1 is normal, this result is valid with the uniform bound on $S(\theta, \tau_2) S(\theta, \tau_1)^{-\nu}$ relaxed to

$$\sup_{0 \leqslant \theta < 2\pi} |S(\theta, \tau_2) S(\theta, \tau_1)^{-\nu}| < (2M_1')^{-1}.$$

PROOF. Lemma 3.7 of Part II implies that if τ_1 is normal then $S(\theta, \tau_1)$ is normal, $0 \leqslant \theta < 2\pi$. The hypotheses, through Theorem 3.4, establish that $e(\theta, \lambda, \tau_1 + \tau_2)$ is one-dimensional for all real θ and $\lambda \in \sigma(S(\theta, \tau_1 + \tau_2))$. Hence the theorem follows from Theorem 3.2. Q.E.D.

The rest of the paper will be concerned with perturbations of operators with constant coefficients. After some preliminaries the central perturbation theorem will be proved as Theorem 3.6. It is a sharpening of the result of Theorem 3.5.

DEFINITION 3.3. For any complex number θ let $D(\theta) = S(\theta, id/dt)$. Let $D = T(id/dt)$.

LEMMA 3.3. *Let $P(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ be a polynomial with constant coefficients. Then*

$$S(\theta, P(id/dt)) = P(D(\theta))$$

and

$$\sigma(S(\theta, P(id/dt))) = \{P(\theta + 2\pi j),\ j = 0, \pm 1, \cdots\}.$$

Furthermore, $\dim e(\theta, \lambda, P(id/dt)) L_2(0, 1)$ *equals the number of times that* λ *appears in the sequence* $\{P(\theta) + 2\pi j), j = 0, \pm 1, \pm 2, \cdots\}$. *If* θ *is real,* $S(\theta, P(id/dt))$ *is a normal operator.*

PROOF. The domain of $P(D(\theta))$ equals the domain of $D(\theta)^n$ equals $\{f \mid D(\theta)^k f \in$ the domain of $D(\theta)$, $k = 0, \cdots n - 1\}$ equals $\{f \mid f^{(k)} \in A^0[0, 1]$, $f^{(k)}(1) = e^{i\theta} f^{(k)}(0)\}$ equals the domain of $S(\theta, P(id/dt)$. Thus $P(D(\theta)) = S(\theta, P(id/dt))$. If θ is real the fact that $S(\theta, P(id/dt))$ is normal can be concluded either from Lemma 3.1 of Part II or from the fact that it is a polynomial in the self-adjoint operator $D(\theta)$. Since $D(\theta)$ is self-adjoint it can have no proper generalized eigenvectors and hence $\dim e(\theta, \lambda, id/dt)$ is just the number of linearly independent solutions of $(id/dt - \lambda) f = 0$ satisfying the boundary conditions defining $D(\theta)$. But there can be at most one such f up to a scalar factor and hence $\dim e(\theta, \lambda, id/dt) L_2(0, 1) = 1$, $\lambda \in \sigma(D(\theta))$. It is easy to see that $\sigma(D(\theta)) = \{(\theta + 2\pi j), j = 0, \pm 1, \pm 2, \cdots\}$ and hence by the spectral mapping theorem

$$\begin{aligned}\sigma(S(\theta, P(id/dt))) &= \sigma(P(D(\theta))) = P(\sigma(D(\theta))) \\ &= \{P(\theta + 2\pi j), j = 0, \pm 1, \pm 2, \cdots\}.\end{aligned}$$

Also by the spectral mapping theorem

$$e\left(\theta, \lambda, P\left(\frac{id}{dt}\right)\right) = \sum_{\sigma} e\left(\theta, \sigma, \frac{id}{dt}\right)$$

where the summation is over all σ such that $P(\sigma) = \lambda$. Since, for $\sigma_1 \neq \sigma_2$, $e(\theta, \sigma_1, id/dt)\, e(\theta, \sigma_2, id/dt) = 0$, and $\dim e(\theta, \sigma, id/dt) L_2(0, 1) = 1$ for $\sigma \in \sigma(D(\theta))$, $\dim e(\theta, \lambda, P(id/dt)) L_2(0, 1)$ equals the number of σ in $\sigma(D(\theta))$ such that $P(\sigma) = \lambda$. Q.E.D.

LEMMA 3.4. *Let* $P(z) = z^n + a_1 z^{n-1} + a_1 z^{n-2} + \cdots + a_n$ *be a polynomial with constant coefficients. Then there is a constant* $M > 0$ *such that for* $|z_1| \geqslant M$ *the inequality* $|z_2| \geqslant 2|z_1|$ *implies*

$$|P(z_2) - P(z_1)| \geqslant \tfrac{1}{2}|z_2^n - z_1^n|.$$

PROOF. Let $Q(z) = P(z) - z^n$, a polynomial of degree at most $n - 1$. Then for $|z|$ sufficiently large, $|Q(z)| \leqslant K|z|^{n-1}$ for some positive and finite K. Hence for $2|z_1| \leqslant |z_2|$ and $|z_1|$ sufficiently large,

$$|Q(z_1)| \leqslant |Q(z_2)| \leqslant K|z_2|^{n-1}.$$

Thus

$$\begin{aligned} |P(z_1) - P(z_2)| &\geqslant |z_1^n - z_2^n| - |Q(z_1)| - |Q(z_2)| \\ &\geqslant |z_1^n - z_2^n| - 2K|z_2|^{n-1}. \end{aligned}$$

Let $z_2 = uz_1$, $|u| \geqslant 2$. Then

$$\frac{z_2^{n-1}}{z_2^n - z_1^n} = \frac{u^{n-1}}{u^n - 1} z_1^{-1}.$$

Since $u^{n-1}(u^n - 1)^{-1}$ is uniformly bounded for $|u| \geqslant 2$, we have

$$|z_2|^{n-1} \leqslant K'|z_1|^{-1}|z_2^n - z_1^n|$$

and thus

$$|P(z_2) - P(z_1)| \leqslant |z_2^n - z_1^n| - K'' \frac{|z_2^n - z_1^n|}{|z_1|}$$

and hence, for $|z_1|$ sufficiently large,

$$|P(z_2) - P(z_1)| \geqslant |z_2^n - z_1^n|/2.$$

Q.E.D.

DEFINITION 3.4. For a polynomial $P(z)$ and any θ, $0 \leqslant \theta < 2\pi$, let $\lambda_j(\theta) = P(\theta + 2\pi j)$, $j = 0, \pm 1, \pm 2, \cdots$. Let $d_j(\theta) = \min_{i \neq j} |\lambda_i(\theta) - \lambda_j(\theta)|$. Let $d_j = \inf_{0 \leqslant \theta < 2\pi} d_j(\theta)$.

Note that $d_j(\theta) = 0$ if and only if $\lambda_i(\theta) = \lambda_j(\theta)$ for some $i \neq j$ and hence if and only if $\lambda_j(\theta)$ is a multiple point in the spectrum of $P(D(\theta))$.

THEOREM 3.6. *Let* $P(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ *be a polynomial with constant coefficients. Let* τ *be a differential operator with periodic coefficients of the form*

$$\tau = P\left(\frac{id}{dt}\right) + a_1(t)\left(\frac{id}{dt}\right)^{n-1} + \cdots + a_n(t)$$

and let

$$|a_j|_\infty = \max_{0 \leqslant t \leqslant 1} |a_j(t)|.$$

(a) *For arbitrary* θ, $0 \leqslant \theta < 2\pi$ *and each integer* k, $0 \leqslant k \leqslant n-1$, *let*

$$M_k(\theta) = \sup_j \frac{|\theta + 2\pi j|^k}{d_j(\theta)} \leqslant \infty.$$

If

$$\sum_{k=0}^{n-1} |a_{n-k}|_\infty M_k(\theta) < \tfrac{1}{2}$$

then $\sigma(S(\theta, \tau))$ *can be enumerated as* $\{u_k(\theta)\}_{-\infty}^{\infty}$ *in such a manner that*

$$|\lambda_k(\theta) - u_k(\theta)| < \frac{d_k(\theta)}{2}$$

and

$$\dim e(\theta, u_k(\theta), \tau) L_2(0, 1) = 1.$$

If

$$\sup_{0 \leqslant \theta < 2\pi} \sum_{k=0}^{n-1} |a_{n-k}|_\infty M_k(\theta) < \tfrac{1}{2}$$

then $T(\tau)$ *is spectral on compact sets.*

(b) *For each integer* k, $0 \leqslant k \leqslant n-2$ *and integer* j, $-\infty < j < \infty$, *let*

$$s_{jk} = \frac{(2\pi |j| + 1)^k}{d_j} \qquad \textit{(which may be infinite)}$$

and suppose that for some integer K, $0 \leqslant K \leqslant n-2$ *and integer* N

$$\sum_{|j| \geqslant N} (s_{jK})^2 < \infty.$$

Then if $a_j(t) \equiv 0$ *for* $j < n - K$, $S(\theta, \tau)$ *is a spectral operator for* $0 \leqslant \theta < 2\pi$ *and* $T(\tau)$ *is spectral at infinity.*

(c) *If* (b) *is satisfied and* (a) *is satisfied for all* θ, $0 \leqslant \theta < 2\pi$ *then* $T(\tau)$ *is a spectral operator of scalar type.*

REMARK. Note that our conditions are such that for (a) no eigenvalue of $P(D(\theta))$ is multiple and for (b) there is a neighborhood of ∞ in which no eigenvalue of $P(D(\theta))$ is multiple, $0 \leqslant \theta < 2\pi$. Also note that if $M_k(\theta) < \infty$

for some k then $M_{k'}(\theta) < \infty$ for $k' < k$. Letting l, $0 \leqslant l \leqslant n-2$ be such that $M_k(\theta) = \infty$, $k > l$, the condition $\sum_{k=0}^{n-1} |a_{n-k}|_\infty M_k(\theta) < \frac{1}{2}$ implies of course that $a_{n-k}(t) = 0$, $k > l$.

Note also that Part (b) which is essentially an asymptotic result is a restriction only on the order of the perturbing operator, not on the size of its coefficients, in contrast to part (a).

PROOF. (a) Using the notation introduced in Theorem 3.3, let

$$Q(\theta) = [a_1]\, D(\theta)^{n-1} + [a_2]\, D(\theta)^{n-2} + \cdots + [a_n]$$

and let $T(\theta) = P(D(\theta))$. We can show using the same argument as in Theorem 3.3 that $S(\theta, \tau) = T(\theta) + Q(\theta)$.

Let $C_j(\theta)$ be the circle about $\lambda_j(\theta)$ of radius $d_j(\theta)/2$ and let $C_j{}^0(\theta)$ be the open disc enclosed by $C_j(\theta)$.

Let $u \in \bigcap_j (C_j{}^0(\theta)')$. If A is a normal operator then

$$| f(A) | = \operatorname*{ess\,sup}_{\lambda \in \sigma(A)} | f(\lambda) |.$$

Hence

$$| D^k(\theta)\, R(u, T(\theta)) | = \sup_j \frac{| \theta + 2\pi j |^k}{| u - \lambda_j(\theta) |} \leqslant 2 M_K(\theta)$$

Hence

$$| Q(\theta)\, R(u, T(\theta)) | \leqslant 2 \sum_k | a_{n-k} |_\infty M_k(\theta) < 1$$

and the first conclusion follows from Lemma 3.2. The second conclusion of (a) then follows from Theorem 3.2.

(b) Let $A = \max_{n-K \leqslant l \leqslant n} | a_l |_\infty$ and let us assume that the constant N above has been chosen so large that

(i) $N \geqslant 2$.

(ii) $\sum_{|j| \geqslant N} (s_{jK})^2 < (4 \cdot 3^K A(K+1))^{-2}$.

(iii) $N \geqslant M$, the constant of Lemma 3.4 for the polynomial $P(z)$.

The finiteness of the sum in (ii) implies that $\lim_{|j| \to \infty} d_j(\theta) = \infty$ so that there exists an integer $j(\theta)$, $| j(\theta) | \geqslant N$ such that $d_{j(\theta)}(\theta) = \min_{|j| \geqslant N} d_j(\theta)$. Let $d_{j(\theta)}(\theta) = d(\theta)$ and let

$$N(\theta) = \left\{ u \;\middle|\; | u - \lambda_j(\theta) | \geqslant \frac{d(\theta)}{2},\ | j | < N \right\}.$$

Let

$$u \in N(\theta) - \bigcup_{|j| \geqslant N} C_j{}^0(\theta)$$

and consider, for $0 \leqslant k \leqslant K$,

$$| D(\theta)^k R(u, T(\theta)) | = \sup_j \frac{| \theta + 2\pi j |^k}{| u - \lambda_j(\theta) |}, \qquad 0 \leqslant k \leqslant K.$$

For all l,

$$2\pi(| l | - 1) \leqslant | \theta + 2\pi l | \leqslant 2\pi(| l | + 1).$$

Hence for $| j | < N$,

$$| \theta + 2\pi j | \leqslant 2\pi(N + 1) \leqslant (N + 1)(N - 1)^{-1} | \theta + 2\pi j(\theta) | \leqslant 3 | \theta + 2\pi j(\theta) |.$$

Also for $| j | < N$,

$$| u - \lambda_j(\theta) | \geqslant \frac{d(\theta)}{2} = \frac{d_{j(\theta)}(\theta)}{2}$$

and thus

$$\begin{aligned} \frac{| \theta + 2\pi_j |^k}{| u - \lambda_j(\theta) |} &\leqslant 2 \cdot 3^k \frac{| \theta + 2\pi j(\theta) |^k}{d_{j(\theta)}(\theta)} \\ &\leqslant 2 \cdot 3^k \frac{| \theta + 2\pi j(\theta) |^K}{d_{j(\theta)}(\theta)}, \qquad | j | < N, \qquad 0 \leqslant k \leqslant K, \\ &\qquad\qquad u \in N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta). \end{aligned}$$

Furthermore

$$\begin{aligned} \frac{| \theta + 2\pi j |^k}{| u - \lambda_j(\theta) |} &\leqslant \frac{| \theta + 2\pi j |^K}{| u - \lambda_j(\theta) |} \\ &\leqslant 2 \frac{| \theta + 2\pi j |^K}{d_j(\theta)}, \qquad | j | \geqslant N, \qquad 0 \leqslant k \leqslant K, \\ &\qquad\qquad u \in N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta). \end{aligned}$$

Thus

$$\begin{aligned} | D(\theta)^k R(u, T(\theta)) | &\leqslant 2 \cdot 3^K \sup_{|j| \geqslant N} s_{jK} \\ &< [2A(K + 1)]^{-1}, \qquad u \in N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta) \end{aligned}$$

so that

$$| Q(\theta) R(u, T(\theta)) | < \tfrac{1}{2}, \qquad u \in N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta). \tag{1}$$

It follows from Lemma 3.2 that

$$N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta) \subseteq \rho(S(\theta, \tau)).$$

Observing that

$$C_j(\theta) \subseteq N(\theta) - \bigcup_{|j| \geqslant N} C_j^0(\theta), \qquad |j| \geqslant N,$$

we also conclude using Lemma 3.2 that $\sigma(S(\theta, \tau)) \cap N(\theta)$ can be enumerated as $\{u_j(\theta)\}_{|j| \geqslant N}$ in such a way that

$$|u_j(\theta) - \lambda_j(\theta)| < \frac{d_j(\theta)}{2} \qquad \text{and} \qquad \dim e_\theta(u_j(\theta), \tau) L_2(0, 1) = 1.$$

Thus

$$\sup_{\delta \subseteq N(\theta)} |e_\theta(\delta, \tau)| = \sup_{J \in \Phi} \Big| \sum_{j \in J} e_\theta(C_j^0(\theta), \tau) \Big|$$

where the supremum on the left is over all bounded Borel subsets of $N(\theta)$ and the supremum on the right is over all $J \in \Phi$ where Φ is the class of all finite subsets of the set of integers $\{j \mid |j| \geqslant N\}$.

We have (see the proof of Lemma 3.2)

$$R(u, S(\theta, \tau)) = R(u, T(\theta)) (I - Q(\theta) R(u(T(\theta)))^{-1}$$

whenever both factors on the right are well defined bounded operators, in particular for $u \in C_j(\theta)$, $|j| \geqslant N$.

For any operator B such that $1 \in \rho(B)$,

$$(I - B)^{-1} = I + B + B^2(I - B)^{-1}.$$

Using this identity with $B = Q(\theta) R(u, T(\theta))$ in the second line below we write

$$\begin{aligned} &R(u, S(\theta, \tau)) - R(u, T(\theta)) \\ &= R(u, T(\theta)) ((I - Q(\theta) R(u, T(\theta)))^{-1} - I) \\ &= R(u, T(\theta)) [Q(\theta) R(u, T(\theta)) + (Q(\theta) R(u, T(\theta)))^2 (I - Q(\theta) R(u, T(\theta)))] \\ &\qquad\qquad u \in C_j(\theta), \qquad |j| \geqslant N. \end{aligned}$$

For convenience let $e(\theta, \cdot, P(id/dt) = e(\theta, \cdot)$. Integrating the equation displayed above around the countour $C_j(\theta)$ and summing over $j \in J$ yields

$$\begin{aligned} \sum_{j \in J} (e(\theta, C_j^0(\theta), \tau) - e(\theta, C_j^0(\theta))) &= \frac{1}{2\pi i} \sum_{j \in J} \int_{C_j(\theta)} R(u, T(\theta)) Q(\theta) R(u, T(\theta)) du \\ &+ \frac{1}{2\pi i} \sum_{j \in J} \int_{C_j(\theta)} R(u, T(\theta)) [Q(\theta) R(u, T(\theta))]^2 (I - Q(\theta) R(u, T(\theta)))^{-1} du. \qquad (2) \end{aligned}$$

We will estimate the second term on the right side of (2) first.

From (1) it follows that

$$|(I - Q(\theta)\, R(u, T(\theta)))^{-1}| < 2, \qquad u \in C_j(\theta).$$

Also

$$| R\, u, T(\theta)) | = \sup_{-\infty < l < \infty} | u - \lambda_l(\theta) |^{-1} = 2d_j(\theta)^{-1}, \qquad u \in C_j(\theta).$$

We have

$$| D(\theta)^k R(u, T(\theta)) | = \sup_{-\infty < l < \infty} \frac{| \theta + 2\pi l |^k}{| u - \lambda_l(\theta) |}.$$

Set

$$N_j(\theta) = \{l \mid | \theta + 2\pi l \leqslant 2 \mid \theta + 2\pi j |\}.$$

Then

$$\max_{l \in N_j(\theta)} \frac{| (\theta + 2\pi l) |^k}{| u - \lambda_l(\theta) |} \leqslant 2^{K+1} \frac{| \theta + 2\pi j |^K}{d_j(\theta)}, \qquad u \in C_j(\theta), \qquad 0 \leqslant k \leqslant K.$$

If $0 \leqslant k \leqslant K$, $u \in C_j(\theta)$, $| j | \geqslant N$, and $l \notin N_j(\theta)$, then

$$\begin{aligned}
\frac{| \theta + 2\pi l |^k}{| u - \lambda_l(\theta) |} &\leqslant \frac{| \theta + 2\pi l |^K}{| u - \lambda_l(\theta) |} \\
&\leqslant \frac{| \theta + 2\pi j |^K}{| u - \lambda_l(\theta) |} + \frac{| (\theta + 2\pi l)^K - (\theta + 2\pi j)^K |}{| u - \lambda_l(\theta) |} \\
&\leqslant \frac{2 | \theta + 2\pi j |^K}{d_j(\theta)} + \frac{| (\theta + 2\pi l)^K - (\theta + 2\pi j)^K |}{| u - \lambda_l(\theta) |}.
\end{aligned}$$

Now

$$\begin{aligned}
| u - \lambda_l(\theta) | &\geqslant | \lambda_j(\theta) - \lambda_l(\theta) | - | \lambda_j(\theta) - u | \\
&= | \lambda_j(\theta) - \lambda_l(\theta) | - \frac{d_j(\theta)}{2} \geqslant \frac{| \lambda_j(\theta) - \lambda_l(\theta) |}{2} \\
&= \frac{| P(\theta + 2\pi j) - P(\theta + 2\pi l) |}{2} \geqslant \frac{| (\theta + 2\pi j)^n - (\theta + 2\pi l)^n |}{4}
\end{aligned}$$

where we have used Lemma 3.4 in the last inequality. Thus for $l \notin N_j(\theta)$, $u \in C_j(\theta)$, $| j | \geqslant N$,

$$\frac{| (\theta + 2\pi l)^K - (\theta + 2\pi j)^K |}{| u - \lambda_l(\theta) |} \leqslant 4 \frac{| \theta + 2\pi l |^K + | \theta + 2\pi j |^K}{| \theta + 2\pi l |^n - | \theta + 2\pi j |^n}.$$

Consider the function

$$g(x) = \frac{x^K + x_1{}^K}{x^n - x_1{}^n}$$

for positive $x \geqslant 2x_1$. Let $v = x/x_1$ and write

$$g(x) = x_1^{K-n} \frac{v^K + 1}{v^n - 1}, \qquad v \geqslant 2.$$

Since

$$\lim_{v\to\infty} (v^K + 1)(v^n - 1)^{-1} = 0$$

there is a constant M depending only on n and K such that $| g(x) | \leqslant M x_1^{K-n}$, $x \geqslant 2x_1 > 0$. Thus

$$\frac{|(\theta + 2\pi l)^K - (\theta - 2\pi j)^K|}{| u - \lambda_l(\theta) |} \leqslant 4M | \theta + 2\pi j |^{K-n},$$

$$l \notin N_j(\theta), \qquad u \in C_j(\theta), \qquad | j | \geqslant N,$$

and thus

$$\frac{| \theta + 2\pi l |^k}{| u - \lambda_l(\theta) |} \leqslant 2 \frac{| \theta + 2\pi j |^K}{d_j(\theta)} + 4M | \theta + 2\pi j |^{K-n},$$

$$l \notin N_j(\theta), \qquad u \in C_j(\theta), \qquad | j | \geqslant N, \qquad 0 \leqslant k \leqslant K.$$

Combining this with the estimate for $l \in N_j(\theta)$ we get

$$| D(\theta)^k R(u, T(\theta)) | \leqslant 2^{K+1} \frac{|\theta - 2\pi j|^K}{d_j(\theta)} + 4M | (\theta + 2\pi j) |^{K-n}, 0 \leqslant k \leqslant K,$$
$$u \in C_j(\theta), | j | \geqslant N,$$

and hence

$$| Q(\theta) R(u, T(\theta)) | \leqslant C(s_{jK} - | \theta + 2\pi j |^{K-n}), \qquad u \in C_j(\theta), \qquad | j | \geqslant N,$$

where C is a constant independent of θ. We have then,

$$\left| \frac{1}{2\pi i} \sum_{j\in J} \int_{C_j(\theta)} R(u, T(\theta)) [Q(\theta) R(u, T(\theta))]^2 (I - Q(\theta) R(u, T(\theta)))^{-1} du \right|$$

$$\leqslant \sum_{j\in J} \frac{d_j(\theta)}{2} \sup_{u\in C_j(\theta)} | R(u, T(\theta)) | \, | Q(\theta) R(u, T(\theta)) |^2 \, | (I - Q(\theta) R(u, T(\theta)))^{-1} |$$

$$\leqslant \sum_{j\in J} \frac{d_j(\theta)}{2} (2 d_j(\theta)^{-1}) C^2 (s_{jK} + | \theta + 2\pi j |^{K-n})^2 \cdot 2$$

$$\leqslant 4C^2 \sum_{j\in J} (s_{jK}^2 + | \theta + 2\pi j |^{2(K-n)})$$

$$\leqslant 4C^2 \sum_{|j|\geqslant N} (s_{jK}^2 + | \theta + 2\pi j |^{2(K-n)})$$

$$\leqslant F$$

where F is a finite constant independent of θ.

We use the method of residues to compute the first term on the right of (2),

$$\frac{1}{2\pi i}\sum_{j\in J}\int_{C_j(\theta)} R(u, T(\theta))\, Q(\theta)\, R(u, T(\theta))\, du$$

$$= \frac{1}{2\pi i}\sum_{j\in J}\sum_{0\leq k\leq K}\int_{C_j(\theta)} R(u, T(\theta))\, [a_{n-k}]\, D(\theta)^k\, R(u, T(\theta))\, du.$$

We have

$$R(u, T(\theta)) = (u - \lambda_j(\theta))^{-1}\, e(\theta, \lambda_j(\theta)) + R_j(u, T(\theta))$$

where

$$R_j(u, T(\theta)) = \sum_{i\neq j}(u - \lambda_i(\theta))^{-1}\, e(\theta, \lambda_i(\theta))$$

which is analytic in u for u in $C_j^0(\theta)$. Thus

$$\begin{aligned} R(u, T(\theta))\, [a_{n-k}]\, D(\theta)^k\, R(u, T(\theta)) &= (u - \lambda_j(\theta))^{-2}\, e(\theta, \lambda_j(\theta))\, [a_{n-k}]\, D(\theta)^k\, e(\theta, \lambda_j(\theta)) \\ &\quad + (u - \lambda_j(\theta))^{-1}\, [e(\theta, \lambda_j(\theta))\, [a_{n-k}]\, D(\theta)^k\, R_j(u, T(\theta)) \\ &\quad + R_j(u, T(\theta))\, [a_{n-k}]\, D(\theta)^k\, e(\theta, \lambda_j(\theta))] + P_j(u, \theta) \end{aligned}$$

where $P_j(u, \theta)$ is analytic in u for u in $C_j^0(\theta)$. Using residues and the fact that

$$D(\theta)^k\, e(\theta, \lambda_j(\theta)) = (\theta + 2\pi j)^k\, e(\theta, \lambda_j(\theta)),$$

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_j(\theta)} R(u, T(\theta))\, Q(\theta)\, R(u, T(\theta))\, du &= e(\theta, \lambda_j(\theta))\, Q(\theta)\, R_j(\lambda_j(\theta), T(\theta)) \\ &\quad + R_j(\lambda_j(\theta), T(\theta)) \sum_{k=0}^{K} (\theta + 2\pi_j)^k\, [a_{n-k}]\, e(\theta, \lambda_j(\theta)). \end{aligned}$$

Since the projections $e(\theta, \lambda_j(\theta))$ are self-adjoint and

$$e(\theta, \lambda_j(\theta))\, e(\theta, \lambda_l(\theta)) = 0, \qquad j \neq l, \qquad |j| \geqslant N, \qquad |l| \geqslant N,$$

$$\begin{aligned} \Big| \sum_{j\in J} e(\theta, \lambda_j(\theta))\, Q(\theta)\, R_j(\lambda_j(\theta), T(\theta)) \Big|^2 &\leqslant \sum_{j\in J} |\, e(\theta, \lambda_j(\theta))\, Q(\theta)\, R_j(\lambda_j(\theta), T(\theta))\, |^2 \\ &\leqslant \sum_{j\in J} |\, Q(\theta)\, R_j(\lambda_j(\theta), T(\theta))\, |^2 \qquad (3) \end{aligned}$$

and, taking adjoints,

$$\left| \sum_{j\in J} R_j(\lambda_j(\theta), T(\theta)) \left[\sum_{k=0}^{K} (\theta + 2\pi j)^k [a_{n-k}]\right] e(\theta, \lambda_j(\theta)) \right|^2$$

$$\leqslant \left| \sum_{j\in J} | e(\theta, \lambda_j(\theta)) \left[\sum_{k=0}^{K} (\theta + 2\pi j)^k [a_{n-k}]\right]^* R_j(\lambda_j(\theta), T(\theta))^* \right|^2$$

$$\leqslant A^2 K \sum_{j\in J} \sum_{k=0}^{K} | (\theta + 2\pi j)^k R_j(\lambda_j(\theta), T(\theta)) |^2. \tag{4}$$

Let $u \in C_j(\theta)$. Then

$$\begin{aligned} | D^k(\theta) R_j(\lambda_j(\theta), T(\theta)) | &= \sup_{l\neq j} \frac{| \theta - 2\pi l |^k}{| \lambda_j(\theta) - \lambda_l(\theta) |} \\ &\leqslant \sup_{l\neq j} \frac{| \theta + 2\pi l |^k}{| \, | \lambda_l(\theta) - u | - | u - \lambda_j(\theta) | \, |} \\ &\leqslant \sup_{l\neq j} 2 \frac{| \theta + 2\pi l |^k}{| \lambda_l(\theta) - u |} \\ &\leqslant \sup_{-\infty < l < \infty} 2 \frac{| \theta + 2\pi l |^k}{| \lambda_l(\theta) - u |} \\ &= 2 \, | D^k(\theta) R(u, T(\theta)) | \end{aligned}$$

a quantity we have already estimated above. It follows, using the estimate, that

$$| Q(\theta) R_j(\lambda_j(\theta), T(\theta)) | \leqslant C'(s_{jK} + | \theta + 2\pi j |^{K-n}), \qquad | j | \geqslant N,$$

for some constant C' independent of θ. Thus, using (3),

$$\left| \sum_{j\in J} e(\theta, \lambda_j(\theta)) Q(\theta) R_j(\lambda_j(\theta), T(\theta)) \right|^2 \leqslant C'' \sum_{|j| \geqslant N} (s_{jK} + | \theta + 2\pi j |^{K-n})^2 \leqslant G \tag{5}$$

where G is a finite constant independent of θ.

Estimating the quantity in (4) is easy. For $| j | \geqslant N$, $0 \leqslant k \leqslant n - K$,

$$\begin{aligned} | (\theta + 2\pi j)^k R_j(\lambda_j(\theta), T(\theta)) | &= | \theta + 2\pi j |^k \sup_{l\neq j} \frac{1}{| \lambda_l(\theta) - \lambda_j(\theta) |} \\ &\leqslant \frac{| \theta + 2\pi j |^K}{d_j(\theta)} \\ &\leqslant S_{jK} \end{aligned}$$

so that

$$\left| \sum_{j\in J} R_j(\lambda_j(\theta), T(\theta)) \left(\sum_{k=0}^{K} (\theta + 2\pi j)^k [a_{n-k}] \right) e(\theta, \lambda_j(\theta)) \right|^2 \leqslant C^2 \sum_{|j| \geqslant N} S_{jK}^2 \leqslant H \tag{6}$$

where H is a constant independent of θ.

We have now shown that the right hand side of (2) is bounded in norm uniformly in $J \in \Phi$ and $\theta \in [0, 2\pi)$ by a constant M and hence

$$\left| \sum_{j\in J} e(\theta, C_j^0(\theta), \tau) \right| \leqslant \left| \sum_{j\in J} e(\theta, C_j^0(\theta)) \right| + M = M + 1$$

so that $M + 1$ is an upper bound for $\sup_\delta | e(\theta, \delta, \tau) |$, δ varying over all bounded Borel sets in $N(\theta)$, $0 \leqslant \theta < 2\pi$. Letting

$$N = \bigcap_{0 \leqslant \theta < 2\pi} N(\theta)$$

it follows that $1 + M$ is an upper bound for $\sup_{\delta,\theta} | e(\theta, \delta, \tau) |$ where δ varies over all bounded Borel subsets of N and θ over the interval $[0,2\pi)$. Since N is easily seen to contain a neighborhood of ∞ we have established that $e(\theta, \cdot, \tau)$, $0 \leqslant \theta < 2\pi$, is uniformly essentially bounded near ∞.

Since $\dim e(\theta, u_j(\theta), \tau) = 1$ for $| j |$ large enough, to show that $S(\theta, \tau)$ is spectral for all θ and that $T(\tau)$ is spectral at ∞ it only remains to show that $e(\theta, Z, \tau) = I$, $0 \leqslant \theta < 2\pi$, where Z is the complex plane. This will be done as in Schwartz's paper [3] by showing that $e(\theta, \infty, \tau)$ has finite dimensional range where $e(\theta, \infty, \tau)$ is defined to be $I - e(\theta, Z, \tau)$. For it then follows from Theorem 3.3 of Part II that $e(\theta, \infty, \tau) = 0$.

For any Borel set δ, $(I - e(\theta, \delta, \tau))\, e(\theta, \infty, \tau) = e(\theta, \infty, \tau)$ and hence the range of $(e(\theta, \infty, \tau))$ belongs to the range of $(I - e(\theta, \delta, \tau))$ so it is sufficient to show that $I - e(\theta, \delta, \tau)$ is finite dimensional for some Borel set δ.

As $n \to \infty$ in (5) and (6) the constants G and H can be seen to approach 0 and thus, letting $\delta_n = \{z \mid | z | \geqslant n\}$,

$$\lim_{n\to\infty} | e(\theta, \delta_n, \tau) - e(\theta, \delta_n) | = 0.$$

Thus

$$\lim_{n\to\infty} | (I - e(\theta, \delta_n, \tau)) - (I - e(\theta, \delta_n)) | = 0.$$

Now the unperturbed operator $P(D(\theta))$ is normal so that

$$I - e(\theta, \delta_n) = e(\theta, \delta_n'),$$

which is finite dimensional and hence $I - e(\theta, \delta_n, \tau)$ is finite dimensional for large enough n which shows that $e(\theta, \infty, \tau) = 0$, $0 \leqslant \theta < 2\pi$. Q.E.D.

LEMMA 3.5. *Let $\{a_j\}$ be a sequence of the form $a_j = \max [b_j; c_j]$ where $0 \leqslant b_j < \infty$ and $0 \leqslant c_j \leqslant \infty$. Suppose that $\sum_j b_j < \infty$. Then $\sum_j a_j < \infty$ if and only if $\sum_j c_j < \infty$.*

Lemmas 3.6 and 3.7 below simplify the application of Theorem 3.6(b) by restricting the range of i over which we take $\min_i | \lambda_i(\theta) - \lambda_j(\theta) |$. It will be seen from these lemmas that the crux of applying Theorem 3.6(b) is whether $P(x)$ for $x > 0$ and $P(x)$ for $x < 0$ are sufficiently far apart near ∞.

LEMMA 3.6. *Let $P(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ be a polynomial of degree $n \geqslant 2$. Let*

$$d_j^+(\theta) = \min_{\substack{i \neq j \\ i \cdot j > 0}} | \lambda_i(\theta) - \lambda_j(\theta) |,$$

$$d_j^-(\theta) = \min_{\substack{i \neq j \\ i \cdot j \leqslant 0}} | \lambda_i(\theta) - \lambda_j(\theta) |, \qquad j = 0, \pm 1, \pm 2, \cdots, 0 \leqslant \theta < 2\pi.$$

Let

$$d_j^+ = \min_{0 \leqslant \theta < 2\pi} d_j^+(\theta), \qquad d_j^- = \min_{0 \leqslant \theta < 2\pi} d_j^-(\theta).$$

Then there is an $N \geqslant 0$ such that

$$\sum_{|j| \geqslant N} \left(\frac{|j|^{n-2}}{d_j^+} \right)^2 < \infty.$$

Using the notation of Theorem 3.6(b), given an integer K, $0 \leqslant K \leqslant n-2$, there exists an $N \geqslant 0$ such that $\sum_{|j| \geqslant N} s_{jK}^2 < \infty$ if and only if there exists an $N \geqslant 0$ such that $\sum_{|j| \geqslant N} (|j|^K / d_j^-)^2 < \infty$.

PROOF. Write $P(z) = P_1(z) + iP_2(z)$ where P_1 and P_2 have real coefficients and P_1 is of degree n. Then

$$| P(\theta + 2\pi i) - P(\theta + 2\pi j) | \geqslant | P_1(\theta + 2\pi i) - P_1(\theta + 2\pi j) |$$

so that if the assertion is true for P_1 it is true for P, thus we assume with no loss of generality that P has real coefficients. We restrict ourselves to the proof for $j > 0$; the proof for $j < 0$ is similar. For $x > 0$ and large enough and for $j > 0$,

$$d_j^+(\theta) = P(\theta + 2\pi j) - P(\theta + 2\pi(j-1)) \geqslant 2\pi P'(2\pi(j-1)).$$

Now

$$P'(x) = nx^{n-1}(1 + x^{-1}O(1)) \geqslant \frac{nx^{n-1}}{2}$$

for x large enough and greater than 0. Thus for j large enough, $j > 0$,

$$d_j^+(\theta) \geqslant n\pi(2\pi j - 1)^{n-1}.$$

A similar inequality holds for $j < 0$. Hence for N large enough,

$$\sum_{|j| \geqslant N} \left(\frac{|j|^{n-2}}{d_j^-}\right)^2 < \infty.$$

To prove the second assertion observe that $d_j = \min [d_j^-; d_j^+]$ and hence

$$s_{jK}^2 = \max \left[\left(\frac{|j|^K}{d_j^-}\right)^2; \left(\frac{|j|^K}{d_j^+}\right)^2\right].$$

Since $|j|^K \leqslant |j|^{n-2}$, the first part of the lemma implies that there is an N large enough so that $\sum_{|j| \geqslant N} (|j|^K/d_j^+)^2 < \infty$. Thus using Lemma 3.5, $\sum_{|j| \geqslant N} s_{jK}^2 < \infty$ if and only if $\sum_{|j| \geqslant N} (|j|^K/d_j^-) < \infty$. Q.E.D.

In applying Theorem 3.6(b), not only is it sufficient to look only at the distance between the positive and negative branches but, in certain cases, only at $P(x)$ and $P(-y)$ for x close to y, as the following lemma proves.

Lemma 3.7. *Let n be even and let*

$$P(z) = z^n + a_2 z^{n-2} + a_3 z^{n-3} + \cdots + a_n,$$

i.e., $a_1 = 0$. For any $\epsilon > 0$ and real x let

$$d^-(\epsilon, x) = \inf_{\{y \mid xy < 0, |x+y| \leqslant \epsilon\}} | P(x) - P(y) |.$$

Then there is an $N \geqslant 0$ such that for $|x| \geqslant N$, $d^-(\epsilon, x) \geqslant \epsilon |x|^{n-1}/2$. For any $\epsilon > 0$, integer j, and $\theta \in [0, 2\pi)$, let

$$d_{j\epsilon}^-(\theta) = \min_{\{l \mid jl < 0, |(2\pi j + \theta) + (2\pi l + \theta)| \leqslant \epsilon\}} | P(2\pi j + \theta) - P(2\pi l + \theta) |$$

if the set of l over which the minimum is taken is nonvacuous. If this set is vacuous let $d_{j\epsilon}^-(\theta) = \infty$. Let $d_{j\epsilon}^- = \inf_{0 \leqslant \theta < 2\pi} d_{j\epsilon}^-(\theta)$. Using the notation of Theorem 3.6(b), given an integer K, $0 \leqslant K \leqslant n-2$, there exists an integer N such that $\sum_{|j| \geqslant N} s_{jK}^2 < \infty$ if and only if there exists an $N \geqslant 0$ such that

$$\sum_{|j| \geqslant N} \left(\frac{|j|^K}{d_{j\epsilon}^-}\right)^2 < \infty.$$

Proof. Suppose y is such that $xy < 0$, $|x + y| \geqslant \epsilon$. We write

$$P(x) - P(y) = x^n - y^n + x^{n-2}F(x) + y^{n-2}G(y)$$

where $|F(x)|$ and $|G(y)|$ are bounded by a constant H for x and y bounded away from 0. Suppose $|y| \geqslant 2|x|$. Then we can conclude from Lemma 3.4 that for $|x|$ large enough,

$$\begin{aligned} |P(x) - P(y)| &\geqslant \tfrac{1}{2}|x^n - y^n| \\ &\geqslant \tfrac{1}{2}(|y|^n - |x|^n) \\ &\geqslant \tfrac{1}{2}(2^n - 1)|x|^n \\ &\geqslant (\epsilon/2)|x|^{n-1}. \end{aligned}$$

Suppose $|y| < 2|x|$. Then for $|x|$ large enough,

$$\begin{aligned} |P(x) - P(y)| &\geqslant |x^n - y^n| - H(|x|^{n-2} + |y|^{n-2}) \\ &\geqslant |x^n - y^n| - (1 + 2^{n-2}) H |x|^{n-2} \\ &= |(x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - y^{n-1})| \\ &\qquad - (1 + 2^{n-2}) H |x|^{n-2} \\ &= |x+y|(|x|^{n-1} + |x|^{n-2}|y| + |x|^{n-3}|y|^2 \\ &\qquad + \cdots + |y|^{n-1}) - (1 + 2^{n-2}) H |X|^{n-2} \\ &\geqslant \epsilon |x|^{n-1} - (1 + 2^{n-2}) H |x|^{n-2} \\ &\geqslant \frac{\epsilon}{2}|x|^{n-1}. \end{aligned}$$

We have thus shown for all values of y such that $xy < 0$ and $|x + y| \geqslant \epsilon$, that $|P(x) - P(y)| \geqslant \epsilon |x|^{n-1}/2$ which establishes the first assertion.

We now prove the second assertion. Let

$$\delta_{j\epsilon}^-(\theta) = \min_{\{l \mid jl < 0, |2\pi(j+l)+2\theta| > \epsilon\}} |P(\theta + 2\pi j) - P(\theta + 2\pi l)|,$$

$$\delta_{j\epsilon}^- = \inf_{0 \leq \theta < 2\pi} \delta_{j\epsilon}^-(\theta).$$

Using the notation of Lemma 3.5, for $|j|$ large enough and any $\epsilon > 0$,

$$d_j^-(\theta) = \min\,[\delta_{j\epsilon}^-(\theta); d_{j\epsilon}^-(\theta)]$$

and hence

$$d_j^- = \inf_{0 \leq \theta < 2\pi} d_j^-(\theta) = \min\,[\delta_{j\epsilon}^-; d_{j\epsilon}^-],$$

so

$$\frac{|j|^K}{d_j^-} = \max\left[\frac{|j|^K}{\delta_{j\epsilon}^-}; \frac{|j|^K}{d_{j\epsilon}^-}\right].$$

Using the first part of this lemma, $\delta_{j\epsilon}^{-}(\theta) \geqslant d^{-}(\epsilon, 2\pi j + \theta) \geqslant \epsilon \mid 2\pi j \mid^{n-1}/2$ for $|j| \geqslant N$ and N large enough, and thus $\sum_{|j| \geqslant N} (|j|^K/\delta_{j\epsilon}^{-})^2 < \infty$. It follows from Lemma 3.5 that $\sum_{|j| \geqslant N} (|j|^K/d_j^{-})^2 < \infty$ if and only if $\sum_{|j| \geqslant N} (|j|^K/d_{j\epsilon}^{-})^2 < \infty$, and thus, using Lemma 3.6, $\sum_{|j| \geqslant N} (s_{jK})^2 < \infty$ if and only if $\sum_{|j| \geqslant N} (|j|^K/d_{j\epsilon}^{-})^2 < \infty$. Q.E.D.

THEOREM. 3.7. *Let*

$$\tau = a_0(t) \left(\frac{id}{dt}\right)^n + a_1(t) \left(\frac{id}{dt}\right)^{n-1} + \cdots + a_n(t)$$

be a regular differential operator with periodic coefficients of period 1 such that $\operatorname{var}_{0 \leqslant t \leqslant 1} \arg a_0(t) = 0$ *and assume* τ *has been normalized so that*

$$\arg a_0(t) = 0, \qquad \int_0^1 a_0(t)^{-1/n}\, dt = 1.$$

Let W be the linear homeomorphism of $L_2(-\infty, \infty)$ onto itself of Theorem 2.1 and let

$$\tau_1 = \left(\frac{id}{dt}\right)^n + ia \left(\frac{id}{dt}\right)^{n-1} + b_2(t) \left(\frac{id}{dt}\right)^{n-2} + \cdots + b_n(t)$$

where $T(\tau_1) = WT(\tau)\, W^{-1}$ and a is the real constant

$$a = \operatorname{Im} \int_0^1 \frac{a_1(t)}{a_0(t)}\, dt.$$

Then $T(\tau)$ is spectral at ∞ if one of the following is true.

(a) *n is odd,*

(b) *n is even and $a \neq 0$,*

(c) *n is even and*

$$\tau_1 = \left(\frac{id}{dt}\right)^n + \sum_{j=1}^{2k} b_j \left(\frac{id}{dt}\right)^{n-j} + b_{2k+1} \left(\frac{id}{dt}\right)^{n-2k-1} + b_{2k+2}(t) \left(\frac{id}{dt}\right)^{n-2k-2} + \cdots + b_n(t)$$

where k is an integer, $0 \leqslant k < n/2$, b_j is a real constant, $1 \leqslant j \leqslant 2k$, and b_{2k+1} is a constant with nonzero imaginary part.

(d) *n is even and*

$$\tau_1 = \left(\frac{id}{dt}\right)^n + \sum_{j=1}^{k} b_{2j} \left(\frac{id}{dt}\right)^{n-2j} + b_{2k+1} \left(\frac{id}{dt}\right)^{n-2k-1} + b_{2k+2}(t) \left(\frac{id}{dt}\right)^{n-2k-2} + \cdots + b_n(t)$$

where b_{2j}, $j = 1, 2, \cdots, k$ *is a constant and* b_{2k+1} *is a constant with nonzero imaginary part.*

PROOF. We will use the notation of Lemmas 3.6 and 3.7. For (a) let $P(z) = z^n + iaz^{n-1}$. Since n is odd, for x real Re $P(x)$ has the same sign as x and thus $d_j^-(\theta) \geqslant 2\pi(|j| - 1)^n$, which implies that $\sum_{|j|\geqslant N}(|j|^{n-2}/d_j^-)^2 < \infty$ for N large enough and the conclusion follows from Theorem 3.6(b) and Lemma 3.6.

Observe that (b) is a special case of (c).

To prove (c) let

$$P(z) = z^n + \sum_{j=1}^{2k} b_j z^{n-j} + b_{2k+1}z^{n-2k-1}$$

and observe that for x real, $\operatorname{Im} P(x) = (\operatorname{Im} b_{2k+1})\, x^{n-2k-1}$. Since $n - 2k - 1$ is odd, if x_1 and x_2 are of opposite sign,

$$\begin{aligned} | P(x_1) - P(x_2) | &\geqslant | \operatorname{Im} (P(x_1) - P(x_2)) | \\ &\geqslant | \operatorname{Im} P(x_1) | = | \operatorname{Im} b_{2k+1} | \, | x_1 |^{n-2k-1} \end{aligned}$$

and thus

$$d_j^- \geqslant 2\pi(|j| - 1)^{n-2k-1} \, | \operatorname{Im} b_{2k+1} |.$$

Thus

$$\sum_{|j|\geqslant N} \left(\frac{|j|^{n-2k-2}}{d_j^-}\right)^2 < \infty$$

and we conclude from Theorem 3.6(b) and Lemma 3.6 that we may perturb $P(id/dt)$ with a periodic operator of order less than or equal to $n - 2k - 2$.

For (d) take $P(z)$ to be

$$P(z) = z^n + \sum_{j=1}^{k} b_{2j} z^{n-2j} + b_{2k+1}z^{n-2k-1}.$$

If we let

$$E(z) = z^n + \sum_{j=1}^{k} b_{2j} z^{n-2j} \qquad \text{and} \qquad O(z) = b_{2k+1}z^{n-2k-1}$$

then $E(z)$ is even, $O(z)$ is odd, and $P(z) = E(z) + O(z)$. For x real and u in the interval $-1 \leqslant u \leqslant 1$, let

$$g(x, u) = P(x) - P(-(x + u)).$$

Then

$$g(x, u) = E(x) - E(x + u) + O(x) + O(x + u).$$

Now

$$\begin{aligned} | E(x) - E(x + u) | &\geqslant | \operatorname{Re} (E(x) - E(x + u)) | \\ &\geqslant | u | \min_{-1\leqslant v\leqslant 1} | \operatorname{Re} E'(x + v) |. \end{aligned}$$

Since $\operatorname{Re} E'(x) = nx^{n-1} + (\text{lower order terms})$, for $|x|$ sufficiently large, $|\operatorname{Re} E'(x)| \geqslant (n/2)|x|^{n-1}$ and thus for $|x|$ sufficiently large

$$\min_{-1 \leqslant v \leqslant 1} |\operatorname{Re} E'(x+v)| \geqslant \left(\frac{n}{2}\right) ||x| - 1|^{n-1} \geqslant a|x|^{n-1}$$

for some $a > 0$, and hence

$$|E(x) - E(x+u)| \geqslant a|x|^{n-1}|u|.$$

Thus for $|x|$ large enough, $-1 \leqslant u \leqslant 1$,

$$\begin{aligned} |g(x,u)| &\geqslant ||E(x) - E(x+u)| - |O(x) + O(x+u)|| \\ &\geqslant a|x|^{n-1}|u| - |b_{2k+1}|(|x|^{n-2k-1} + |x+1|^{n-2k-1}) \\ &\geqslant a|x|^{n-1}|u| - b|x|^{n-2k-1} \end{aligned} \tag{7}$$

for some $b > 0$.

Now $\operatorname{Im} E(x)$ is a polynomial of degree at most $n-2$ and hence $\operatorname{Im} E'(x)$ is of degree at most $n-3$. Thus for $|x|$ large enough, $-1 \leqslant u \leqslant 1$,

$$|\operatorname{Im} E(x) - \operatorname{Im} E(x+u)| \leqslant |u| \max_{-1 \leqslant v \leqslant 1} |\operatorname{Im} E'(x+v)| \leqslant c|x|^{n-3}|u|$$

for some constant $c > 0$. By hypothesis, $\operatorname{Im} b_{2k+1} \neq 0$ and thus for $-1 \leqslant u \leqslant 1$,

$$\begin{aligned} |\operatorname{Im}(0(x) + 0(x+u))| &= |\operatorname{Im} b_{2k+1}|(|x|^{n-2k-1} + |x+u|^{n-2k-1}) \\ &\geqslant d|x|^{n-2k-1} \end{aligned}$$

for some $d > 0$ and $|x|$ large enough. Hence

$$\begin{aligned} |g(x,u)| &\geqslant |\operatorname{Im} g(x,u)| \\ &\geqslant |\operatorname{Im}(0(x) + 0(x+u))| - |\operatorname{Im}(E(x) - E(x+u))| \\ &\geqslant d|x|^{n-2k-1} - c|x|^{n-3}|u|. \end{aligned} \tag{8}$$

Combining (7) and (8) we get, for $|x|$ large enough,

$$|g(x,u)| \geqslant \max[a|x|^{n-1}|u| - b|x|^{n-2k-1}, d|x|^{n-2k-1} - c|x|^{n-3}|u|]$$

and thus

$$\min_{-1 \leqslant u \leqslant 1} |g(x,u)| \geqslant \min_{-1 \leqslant u \leqslant 1} \max[a|x|^{n-1}|u| - b|x|^{n-2k-1}, \\ d|x|^{n-2k-1} - c|x|^{n-3}|u|].$$

The two terms on the right are linear in $|u|$, one with positive slope the other with negative slope. They intersect at

$$|u| = \frac{(d+b)|x|^{n-2k+1}}{(a|x|^{n-1} + c|x|^{n-3})}$$

which, for $|x|$ large enough, satisfies $0 < |u| < 1$, and hence for $|x|$ large enough the minimum over $-1 \leqslant u \leqslant 1$ of the maximum will be the common value of the two terms for this value of $|u|$. Thus

$$\min_{-1 \leqslant u \leqslant 1} |g(x, u)| \geqslant a|x|^{n-1} \frac{(d+b)|x|^{n-2k-1}}{a|x|^{n-1} + c|x|^{n-3}} - b|x|^{n-2k-1}$$

$$= |x|^{n-2k-1}\left(\frac{ad - bc|x|^{-2}}{a + c|x|^{-2}}\right).$$

Since a and d are greater than 0, it follows that for $|x|$ large enough

$$\min_{-1 \leqslant u \leqslant 1} |g(x, u)| \geqslant K|x|^{n-2k-1}$$

for some $K > 0$. Using the notation $d_{j\epsilon}^{-}(\theta)$ of Lemma 3.7 and setting $\epsilon = 1$, we have, for $|j|$ large enough,

$$\begin{aligned}
d_{j1}^{-}(\theta) &= \inf_{\{l \mid jl<0, |(2\pi j+\theta)+(2\pi+l+\theta)|<1\}} |P(2\pi j + \theta) - P(2\pi l + \theta)|, \qquad l \text{ integer} \\
&\geqslant \inf_{\{w \mid wj<0, |2\pi j+\theta+w|<1\}} |P(2\pi j + \theta) - P(w)|, \qquad w \text{ real}, \\
&= \min_{-1 \leqslant u \leqslant 1} |g(2\pi j + \theta, u)| \\
&\geqslant K|2\pi j + \theta|^{n-2k-1} \\
&\geqslant K'|j|^{n-2k-1}
\end{aligned}$$

and hence there is an N large enough that

$$\sum_{|j|>N}\left(\frac{|j|^{n-2k-2}}{d_{j1}^{-}}\right)^2 \leqslant K' \sum_{|j|\geqslant N}\left(\frac{|j|^{2-nk-2}}{|j|^{n-2k-1}}\right)^2 = K' \sum_{|j|\geqslant N} |j|^{-2} < \infty.$$

Assertion (d) now follows from Lemma 3.7 and Theorem 3.6(b). Q.E.D.

If n is even and if $P(z) = z^n + iaz^{n-1} + a_2z^{n-2} + \cdots + a_{n-1}z$ is either real or even then the locii $\{P(x) \mid x > 0\}$, $\{P(x) \mid x < 0\}$ are coincident in a neighborhood of ∞. In a neighborhood of ∞, as θ increases from 0 to 2π the points $\{P(\theta + 2\pi j), j > 0\}$ move "out" along the locus and the points $\{P(\theta + 2\pi j), j < 0\}$ move "in" so that $\lim_{|j|\to\infty} d_j = 0$ and Theorem 3.6(b) (and of course Theorem 3.6(a)) yields no information about perturbations of

$P(id/dt)$. It follows from this that Theorem 3.7 has exhausted the applications of Theorem 3.6(b) for n odd and $n = 2$. We have made no vigorous attempt to develop more delicate arguments in order to apply Theorem 3.6(b) to a wider class of polynomials $P(z)$ for n even, $n \geqslant 4$ but the results of Theorem 3.7 can probably be extended for such n.

In the following section we loop at the case $n = 2$ rather carefully, in the process extracting as much as we can from Theorem 3.6(a). We have not pursued the problem of estimating the quantities $M_k(\theta)$ of Theorem 3.6(a) for $n > 2$, feeling that such results would not reveal much more concerning the qualitative nature of the perturbation results and would be of more interest and value if studied in connection with particular operators of special significance.

4. Perturbation Results, Second Order Case

In this section we look at the case of second order differential operators.

We first develop the application of Theorem 3.6 to this case. Using the notation of Theorem 3.6, to get a simple result we do not attempt to find the largest perturbations allowable under the condition

$$\sup_\theta \,[| a_2 |_\infty M_0(\theta) + | a_1 |_\infty M_1(\theta)] < \tfrac{1}{2},$$

but instead look at the more restrictive condition

$$| a_2 |_\infty \sup_\theta M_0(\theta) + | a_1 |_\infty \sup_\theta M_1(\theta) < \tfrac{1}{2}.$$

The quantities $\sup_\theta M_0(\theta)$ and $\sup_\theta M_1(\theta)$ are found in Lemma 4.1 below. These are then applied in a straightforward fashion in Theorems 4.2 and 4.3. Theorem 4.3 employs a change of variable where Theorem 4.2 does not.

By looking at the dependence of

$$\tau = \left(\frac{id}{dt}\right)^2 + 2ia\left(\frac{id}{dt}\right) + q(t)$$

on the real parameter a, interesting qualitative results are obtained. These are discussed and then developed after Theorem 4.3.

Theorem 4.1. *Let*

$$\tau = a_0(t)\left(\frac{id}{dt}\right)^2 + a_1(t)\left(\frac{id}{dt}\right) + a_2(t)$$

where $a_0(t) > 0$. *If*

$$\operatorname{Im} \int_0^1 a_1(t)\, a_0(t)^{-1}\, dt \neq 0$$

then $T(\tau)$ *is spectral at* ∞.

PROOF. This is a direct consequence of Theorem 3.7(b).

LEMMA 4.1. *Let* $P(z) = z^2 + b_1 z + b_2$ *and let* $M_k(\theta)$, $k = 0, 1$, $0 \leqslant \theta < 2\pi$, *be as defined in Theorem 3.6(a). Let*

$$M_k = \sup_{0 \leqslant \theta < 2\pi} M_k(\theta), \qquad k = 0, 1.$$

Then

$$M_0 = | 2\pi \operatorname{Im}(b_1) |^{-1},$$

$$M_1 = \frac{| \, | \operatorname{Re}(b_1) | + 2\pi + i \operatorname{Im}(b_1) |}{4\pi | \operatorname{Im}(b_1) |}.$$

PROOF. For $k = 0, 1$,

$$M_k = \sup_{0 \leqslant \theta < 2\pi} \sup_j \sup_{l \neq j} \frac{| \theta + 2\pi j |^k}{| P(\theta + 2\pi l) - P(\theta + 2\pi j) |} \; l, j \text{ integral}$$

$$= \sup_{0 \leqslant \theta < 2\pi} \sup_j \sup_{m \neq 0} \frac{| \theta + 2\pi j |^k}{| P(\theta + 2\pi j + 2\pi m) - P(\theta + 2\pi j) |} \; m, j \text{ integral}$$

$$= \sup_{-\infty < x < \infty} \sup_{m \neq 0} \frac{| x |^k}{| P(x + 2\pi m) - P(x) |} \; m \text{ integral}$$

$$= \sup_{m \neq 0} \sup_{-\infty < x < \infty} \frac{| x |^k}{2\pi | m | \, | 2x + b_1 + 2\pi m |} \; m \text{ integral.}$$

Letting $k = 0$,

$$M_0 = \sup_{m \neq 0} \sup_{-\infty < x < \infty} \frac{1}{2\pi | m | \, | 2x + b_1 + 2\pi m |} \; m \text{ integral}$$

$$= \frac{1}{2\pi | \operatorname{Im}(b_1) |}.$$

Letting $k = 1$,

$$M_1 = \sup_{m \neq 0} \sup_{-\infty < x < \infty} \frac{| x |}{2\pi | m | \, | 2x + b_1 + 2\pi m |} \; m \text{ integral.}$$

Letting $y = x^{-1}$,

$$\sup_{-\infty < x < \infty} \frac{| x |}{| 2x + b_1 + 2\pi m |} = \sup_{-\infty < y < \infty} \frac{1}{| 2 + y(b_1 + 2\pi m) |}$$

$$= \frac{1}{\inf_{-\infty < y < \infty} | 2 + y(b_1 + 2\pi m) |}.$$

Now

$$\inf_{-\infty<y<\infty} | 2 + y(b_1 + 2\pi m) |^2$$
$$= \inf_{-\infty<y<\infty} [(2 + y(2\pi m + \operatorname{Re} b_1))^2 + y^2(\operatorname{Im} b_1)^2].$$

If $\operatorname{Im}(b_1) = 0$, the infinum is 0 and hence $M_1 = \infty$ which is what the asserted equation for M_1 yields in this case. If $\operatorname{Im}(b_1) \neq 0$, the infinum above can readily be shown to be

$$\inf_{-\infty<y<\infty} | 2 + y(b_1 + 2\pi m) | = \left| \frac{2 \operatorname{Im}(b_1)}{\bar{b}_1 + 2\pi m} \right|$$

and thus

$$M_1 = \sup_{m\neq 0} \frac{| \bar{b}_1 + 2\pi m |}{4\pi | m | | \operatorname{Im}(b_1) |} \qquad m \text{ integral}$$
$$= \sup_{m\neq 0} \frac{| (b_1/2\pi m) + 1 |}{2 | \operatorname{Im}(b_1) |} \qquad m \text{ integral.}$$

Now the points $b_1(2\pi m)^{-1} + 1$ lie on the line $b_1 u + 1$, u real. The supremum over m will occur when $m = \pm 1$, according as $\operatorname{Re} b_1 > 0$ or $\leqslant 0$. From this the stated value of M_1 follows. Q.E.D.

THEOREM 4.2. *Let*

$$\tau = \left(\frac{id}{dt}\right)^2 + b_1 \left(\frac{id}{dt}\right) + b_2 + a_1(t) \left(\frac{id}{dt}\right) + b_2(t)$$

where b_1 and b_2 are constants. If

$$| a_1(\cdot) |_\infty \frac{|| \operatorname{Re}(a_1) | + 2\pi + i \operatorname{Im}(b_1) |}{4\pi | \operatorname{Im}(b_1) |} + | a_2(\cdot) |_\infty \cdot \frac{1}{2\pi | \operatorname{Im}(b_1) |} < \tfrac{1}{2} \tag{1}$$

then $T(\tau)$ is a spectral operator of scalar type.

PROOF. If $\operatorname{Im}(b_1) = 0$ then (1) implies that $a_1(t) \equiv a_2(t) \equiv 0$ so that $T(\tau)$ is normal, hence spectral and of scalar type.

If $\operatorname{Im}(b_1) \neq 0$, (1) implies

$$| a_1(\cdot) |_\infty < \frac{2\pi | \operatorname{Im}(b_1) |}{| | \operatorname{Re}(b_1) | + 2\pi + i \operatorname{Im}(b_1) |}$$
$$< | \operatorname{Im}(b_1) |$$

and thus

$$\left| \int_0^1 (\operatorname{Im}(b_1) + \operatorname{Im}(a_1)(t))\,dt \right| = \left| \operatorname{Im}(b_1) + \int_0^1 \operatorname{Im}(a_1)(t)\,dt \right|$$
$$\geqslant | \operatorname{Im}(b_1) | - \left| \int_0^1 a_1(t)\,dt \right|$$
$$\geqslant | \operatorname{Im}(b_1) | - | a_1(\cdot) |_\infty$$
$$> 0$$

hence

$$\int_0^1 \operatorname{Im}(b_1 + a_1(t))\,dt \neq 0$$

so, by Theorem 4.1, $T(\tau)$ is spectral at ∞. That $T(\tau)$ is spectral on compact sets is a direct consequence of Lemma 4.1 and Theorem 3.6(a). The conclusion is then a consequence of Theorem 3.6(c). Q.E.D.

THEOREM 4.3. *Let*

$$\tau = \left(\frac{id}{dt}\right)^2 + a_1(t)\left(\frac{id}{dt}\right) + a_2(t)$$

and let $a = \int_0^1 \operatorname{Im}(a_1)(t)\,dt$. *If* $a \neq 0$ *and there is a constant* c *such that*

$$\left| a_2(\cdot) - \frac{a_1(\cdot)^2}{4} - \frac{ia_1'(\cdot)}{2} - c \right|_\infty < \pi \mid a \mid$$

then $T(\tau)$ *is a spectral operator of scalar type.*

PROOF. Theorem 4.1 tells us from $a \neq 0$ that $T(\tau)$ is spectral at ∞. Using the automorphism $(Vf)(t) = e^{b(t)}f(t)$ where

$$b(t) = -i2^{-1} \int_0^t a_1(t)\,dt - 2^{-1}at, \qquad T(\tau) = V^{-1}T(\tau_1)\,V$$

where

$$\tau_1 = \left(\frac{id}{dt}\right)^2 + ia\left(\frac{id}{dt}\right) + a_2(t) - \frac{a_1(t)^2}{4} - \frac{ia_1'(t)}{2} - \frac{a^2}{4}.$$

Suppose that a constant c satisfying the hypothesis exists and let

$$\tau_2 = \tau_1 + \frac{a^2}{4} - c.$$

Since by hypothesis

$$\frac{| a_2(\cdot) - \frac{1}{4}a_1(\cdot)^2 - \frac{1}{2}ia_1'(\cdot) - c |_\infty}{2\pi \mid a \mid} < \tfrac{1}{2}$$

we know from Lemma 4.1 and Theorem 3.6(a) that $T(\tau_2)$ and hence $T(\tau)$ is spectral on compact sets. Thus by Theorem 3.6(c), $T(\tau)$ is a spectral operator of scalar type. Q.E.D.

Interesting results can be found if one considers the dependence of the operator $\tau = (id/dt)^2 + 2ia(id/dt) + q(t)$, a real, on a. An inessential translation will simplify the discussion so we make the following definition.

DEFINITION 4.1. *Let*

$$\tau_a = \left(\frac{id}{dt}\right)^2 + 2ia\left(\frac{id}{dt}\right) + q(t) - a^2$$

where a is a real constant.

LEMMA 4.2. *For any θ, $\sigma(S(\theta, \tau_a)) = \sigma(S(\theta - ia, \tau_0))$. The function f is an eigenfunction of $S(\theta, \tau_a) - \lambda I$ if and only if $e^{at}f(t)$ is an eigenfunction of $S(\theta - ia, \tau_0) - \lambda I$.*

PROOF. This follows from the readily verified fact that

$$\tau_a f(t) = e^{-at}\tau_0 e^{at} f(t)$$

and the boundary conditions defining $S(\theta, \tau)$. Q.E.D.

Let $X(t, \lambda) = P(t, \lambda)\, e^{R(\lambda)t}$, $P(t, \lambda)$ periodic in t, be a nonsingular solution of $\dot{X} = A(\tau_0 - \lambda)\, X$. Then (see Coddington and Levinson [7, p. 28])

$$\det X(t, \lambda) = \det X(0, \lambda) \exp \int_0^t \operatorname{trace} A(\tau_0 - \lambda)(t)dt.$$

Now

$$A(\tau_0 - \lambda) = \begin{pmatrix} 0 & 1 \\ q(t) - \lambda & 0 \end{pmatrix}$$

so $\operatorname{tr} A(\tau_0 - \lambda) = 0$. Thus, letting $t = 1$, $\det X(1, \lambda) = \det X(0, \lambda)$, and hence $\det e^{R(\lambda)} = 1$. Let $B(\lambda) = e^{R(\lambda)}$, and denote by $v_i(\lambda)$, $i = 1, 2$, the characteristic values of $\tau - \lambda$ (the eigenvalues of $B(\lambda)$). Let $i\theta_i(\lambda)$, $i = 1, 2$, be the characteristic exponents of $\tau - \lambda$ (the eigenvalues of $R(\lambda)$) which are uniquely determined mod $2\pi i$. The condition $\det B(\lambda) = 1$ means $v_1(\lambda)\, v_2(\lambda) = 1$, or equivalently, $\theta_1(\lambda) + \theta_2(\lambda) \equiv 0 \bmod 2\pi$.

The complex number λ will belong to $\sigma(\tau_a)$ if and only if $\tau_a - \lambda$ has a characteristic value of modulus one, or equivalently, if and only if $\tau_0 - \lambda$ has a characteristic value of modulus $e^{|a|}$. Thus $\sigma(\tau_a)$ is just the set of level curves $|v_i(\lambda)| = e^{\pm a}$, $i = 1, 2$. (Note that if $|v_1(\lambda)| = e^a$ then $|v_2(\lambda)| = e^{-a}$, and vice versa.)

Now $B(\lambda)$ is nonsingular and is an analytic function of λ. Thus if $v_1(\lambda) \neq v_2(\lambda)$ there is a neighborhood of λ in which v_1 and v_2 are analytic functions. If $v_1(\lambda) = v_2(\lambda)$ we know that as σ approaches λ both

$v_1(\sigma)$ and $v_2(\sigma)$ approach the common value of $v_i(\lambda)$, $i = 1, 2$. Since $v_1(\lambda)\, v_2(\lambda) = 1$, we have $v_1(\lambda) = v_2(\lambda)$ if and only if $v_1(\lambda) = \pm 1$ or equivalently $\theta_1(\lambda) = \theta_2(\lambda) = 0$ or π, mod 2π. Since $B(\lambda)$ is nonsingular, $v_i(\lambda) \neq 0$, $i = 1, 2$. The $v_i(\lambda)$ are not identically constant for if they were Lemma 4.2 would imply that $\sigma(S(\theta, \tau_a)) = Z$ for that real θ and a satisfying $i\theta + a = \log v_i(\lambda)$, contradicting the discreteness of $S(\theta, \tau_a)$.

Elementary properties of analytic functions yield some qualitative results of interest which we present in Theorems 4.5-4.7 below. In particular Theorem 4.6 contains a necessary and sufficient condition for $T(\tau_a)$ to be spectral for *all* real $a \neq 0$. Like the theorems of the preceding section these results yield no information as to whether or not $T(\tau_0)$ is spectral.

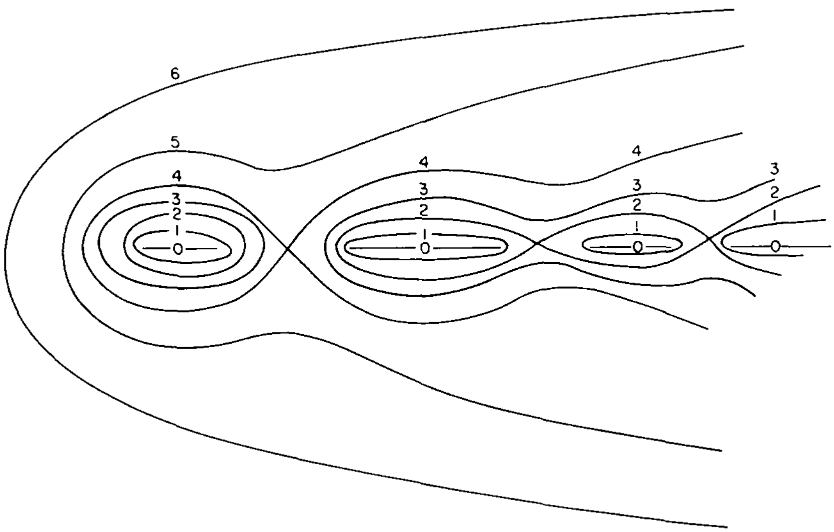

FIG. 1. The spectra of τ_a for different values of a.

Figure 1 illustrates the behavior of $\sigma(\tau_a)$ as a decreases to 0. The figure is for a hypothetical case where $\sigma(\tau_0)$ has several connected components. The labels $0, \cdots, 6$ indicate $\sigma(\tau_a)$ for 7 increasing values of a starting with $a = 0$. For large enough a, the perturbation theory already developed shows that $\sigma(\tau_a)$ is a single simple arc. As a decreases to 0 this arc collapses down to $\sigma(\tau_0)$. In the process the connected components of $\sigma(\tau_0)$ are "pinched off." In this pinching off there will occur values of a for which $\sigma(\tau_a)$ crosses itself (curves 2 and 4 are examples) and for these values of a, $T(\tau_a)$ will not be spectral. The number of $a > 0$ for which $T(\tau_a)$ is not spectral will be less than or equal to the number of bounded connected components of $\sigma(\tau_0)$.

Since, however, for any value of $a > 0$, $\sigma(\tau_a)$ must not cross itself near ∞, $\sigma(\tau_a)$ will encircle only a finite number of components of $\sigma(\tau_0)$ for any $a > 0$. Hence if there are an infinite number of components of $\sigma(\tau_0)$ there will be an infinite but countable number of $a > 0$ for which $\sigma(\tau_a)$ will cross itself and for these a, $T(\tau_a)$ will not be spectral. These statements will be elaborated and justified in the following pages.

LEMMA 4.3. *If $\lambda \in \sigma(S(\theta, \tau_0))$ then $\lambda \in \sigma(S(\phi, \tau_0))$ if and only if*

$$\theta \pm \phi \equiv 0 \bmod 2\pi.$$

PROOF. A point $\lambda \in \sigma(S(\phi, \tau_0))$ if and only if $(\tau_0 - \lambda) f = 0$ has a solution of the form $p(t)\, e^{i\phi t}$, $p(t)$ periodic. But this means that $e^{i\phi}$ is an eigenvalue of $B(\lambda)$, i.e., either $\phi \equiv \theta \bmod 2\pi$ or $\theta + \phi \equiv 0 \bmod 2\pi$. Q.E.D.

LEMMA 4.4. *Let $\lambda \in \sigma(S(\theta, \tau_a))$. If $\theta - ia \equiv 0$ or $\pi \bmod 2\pi$ there will be only one linearly independent solution of $(S(\theta, \tau_a) - \lambda I) f = 0$.*

PROOF. Using Lemma 4.2 we can assume without loss of generality that $a = 0$. If there are two linearly independent solutions of $(S(\theta, \tau_0) - \lambda I) f = 0$ then $\theta_i(\lambda) \equiv \theta \bmod 2\pi$, for both $i = 1, 2$. But $\theta_1(\lambda) \equiv -\theta_2(\lambda) \bmod 2\pi$ hence this can be only if $\theta \equiv 0$ or $\pi \bmod 2\pi$. Q.E.D.

COROLLARY 4.1. *If $0 \leqslant \theta < 2\pi$ and $a \neq 0$ or if $a = 0$ and both $0 < \theta < 2\pi$ and $\theta \neq \pi$, then* $\dim e(\theta, \lambda, \tau_a) L_2(0, 1) > 1$ *if and only if $S(\theta, \tau_a) \mid e(\theta, \lambda, \tau_a) L_2(0, 1)$ is not scalar type.*

PROOF. By Lemma 4.4, $(S(\theta, \tau_a) - \lambda I) f = 0$ for at most one f, hence if $\dim e(\theta, \lambda, \tau_a) L_2(0, 1) > 1$ there is a proper generalized eigenvector of $S(\theta, \tau_a)$ for the eigenvalue λ. Q.E.D.

The following theorem improves Lemma 3.1. It relates the analytic behavior of the functions $v_i(\lambda)$ to the boundedness of $e(\theta, \cdot, \tau_a)$.

THEOREM 4.4. *Let $\lambda_0 \in \sigma(S(\theta_0, \tau_a))$ where $0 \leqslant \theta_0 < 2\pi$. If $a = 0$ assume also that $\theta_0 \neq 0$ or π. A necessary and sufficient condition that $e(\theta, \cdot, \tau_a)$, $0 \leqslant \theta < 2\pi$, be uniformly essentially bounded near λ_0 is that*

$$\dim (e(\theta_0, \lambda_0, \tau_a) L_2(0, 1)) = 1.$$

An equivalent necessary and sufficient condition is that $\sigma(\tau_a)$ be homeomorphic to $(-1, 1)$ in some neighborhood of λ_0.

PROOF. The assumptions on a and θ_0 are such that $\theta_1(\lambda)$ and $\theta_2(\lambda)$ are analytic near λ_0. Suppose that $\theta_1(\lambda)$ and $\theta_2(\lambda)$ have been so labeled that $\theta_1(\lambda_0) = \theta_0$. Since by Corollary 4.1 it is impossible for

$$\dim e(\theta_0, \lambda_0, \tau_a) L_2(0, 1) > 1$$

without $S(\theta_0, \tau_a) \mid e(\theta_0, \lambda_0, \tau_a) L_2(0, 1)$ being nonscalar, Lemma 3.1 for this case becomes a necessary and sufficient statement, i.e., there exists a neighborhood V of θ_0 such that $e(\theta, \cdot, \tau_a)$, $\theta \in V$, is uniformly bounded near λ_0 if and only if

$$\dim e(\theta_0, \lambda_0, \tau_a) L_2(0, 1) = 1.$$

Let $\lambda = \lambda(\theta_1)$ be the (possibly multiple valued) inverse function of $\theta_1(\lambda)$. If $\dim e(\theta_0, \lambda_0, \tau_a) L_2(0, 1) = 1$ then $\lambda(\theta_1)$ is single valued and analytic near θ_0 so that $\theta_1'(\lambda_0) = (\lambda'(\theta_0))^{-1} \neq 0$ and hence

$$\{\lambda \mid \lambda = \lambda(\theta_1), \theta_1 \text{ real}\} = \{\lambda \mid \theta_1(\lambda) \text{ is real}\}$$

is homeomorphic to $(-1, 1)$ in the vicinity of λ_0. Conversely, suppose $\{\lambda \mid \theta_1(\lambda) \text{ is real}\} \cap N$ is homeomorphic to $(-1, 1)$ for some neighborhood N of λ_0. Then $\theta_1'(\lambda_0) \neq 0$ and $\lambda(\theta_1)$ is a single valued function of θ, for θ_1 in a neighborhood of θ_0. But if $\dim e(\theta_0, \lambda_0, \tau_a) L_2(0, 1)$ were to be > 1 then by Lemma 3.12 of Part II λ_0 would be a branch point of $\sigma(S(\theta_0, \tau_a))$ and hence $\lambda(\theta_1)$ could not be single valued near θ_0. Thus $\dim e(\theta_0, \lambda_0, \tau_a) L_2(0, 1) = 1$ if and only if there is a neighborhood of λ_0 in which $\sigma(\tau_a)$ is homeomorphic to $(-1, 1)$.

To complete the proof it is only necessary to show that if there is a neighborhood V of θ_0 such that $e(\theta, \cdot, \tau_a)$, $\theta \in V$, is uniformly essentially bounded near λ_0 then $e(\theta, \cdot, \tau_a)$, $0 \leq \theta < 2\pi$ is uniformly essentially bounded near λ_0. If $a \neq 0$ then $\theta \neq \theta_0$, $0 \leqslant \theta < 2\pi$ implies $\lambda \notin \sigma(S(\theta, \tau_a))$ by Lemma 4.3 and thus there is a neighborhood of λ_0 such that for λ in this neighborhood and θ not in V, $e(\theta, \lambda, \tau_a) = 0$ so the assertion follows for $a \neq 0$. If $a = 0$ then $\theta \neq \theta_0$, $0 \leqslant \theta < 2\pi$, implies $\lambda_0 \notin \sigma(S(\theta, \tau_a))$ by Lemma 4.3 *except* for $\theta \equiv -\theta_0$, mod 2π, i.e., $\theta \equiv \theta_2(\lambda_0) \bmod 2\pi$. But since $\theta_1(\lambda) + \theta_2(\lambda) \equiv 0 \bmod 2\pi$, $\theta_1'(\lambda_0) \neq 0$ if and only if $\theta_2'(\lambda_0) \neq 0$ and one can conclude from the above that there exists a neighborhood V^* of $\theta \equiv \theta_2(\lambda_0) \bmod 2\pi$, $0 \leqslant \theta < 2\pi$, such that $\{e(\theta, \cdot, \tau_0), \theta \in V^*\}$, is uniformly essentially bounded near λ_0. Since for $\theta \notin V \cup V^*$ one has $\lambda_0 \in \rho(S(\theta, \tau_a))$, the desired conclusion can be reached in this case also. Q.E.D.

As a consequence of Lemma 4.2,

$$\sigma(\tau_a) = \bigcup_{0 \leqslant \theta < 2\pi} \sigma\,(S(\theta, \tau_a)) = \bigcup_{0 \leqslant \theta < 2\pi} \sigma\,(S(\theta - ia, \tau_0))$$

$$= \{\lambda \mid \operatorname{Im} \theta_i(\lambda) = -a, i = 1 \text{ or } 2\} = \{\lambda \mid | v_i(\lambda) | = e^a, i = 1 \text{ or } 2\}.$$

Since $v_1(\lambda)\, v_2(\lambda) = 1$, if $\lambda \in \sigma(\tau_a)$ then one of the characteristic values of $\tau_0 - \lambda$ has absolute value e^a and the other has absolute value e^{-a}. For $\lambda \notin \sigma(\tau_a)$ either both $v_i(\lambda)$, $i = 1, 2$, lie inside the ring $\{z \mid e^{-|a|} < | z | < e^{|a|}\}$ or both lie outside the closure of this ring.

DEFINITION 4.2. For λ in the complex plane, let $w(\lambda) = \max_{i=1,2} | v_i(\lambda) |$. For real a, let $S(a, +)$ be the set of all λ such that $w(\lambda) > e^{|a|}$. Let $S(a, -)$ be the set of all λ such that $w(\lambda) < e^{|a|}$. For $a \neq 0$ and real let $n(a)$ be the number of connected components of $S(a, -)$.

Note that $w(\lambda)$ is uniquely defined and everywhere continuous. Hence the sets $S(a, \pm)$ are open and if $| a_1 | > | a_2 |$ then

$$S(a_1, +) \subset S(a_2, +), \qquad S(a_1, -) \supset S(a_2, -).$$

LEMMA 4.5. *Let A and B be subsets of the complex plane such that A is open and B is connected. If $B \cap (\bar{A} - A) = \phi$ then either $B \cap A = \phi$ or $B \subseteq A$.*

PROOF. By hypothesis we have $B = (B \cap A) \cup (B \cap \bar{A}')$. If the assertion is false then both sets in this decomposition are nonempty. But this contradicts the connectedness of B. Q.E.D.

THEOREM 4.5. *In the following statements a is always real.*

(a) *The complex plane has the disjoint decomposition*

$$Z = \sigma(\tau_a) \cup S(a, +) \cup S(a, -).$$

(b) *If $a \neq 0$ then for every $\lambda \in \sigma(\tau_a)$* and $\epsilon > 0$,

$$\{\sigma \mid | \lambda - \sigma | < \epsilon\} \cap S(a, +) \neq \phi, \qquad \{\sigma \mid | \lambda - \sigma | < \epsilon\} \cap S(a, -) \neq \phi.$$

(c) *Every connected subset of $\rho(\tau_a)$ lies either in $S(a, +)$ or $S(a, -)$.*

(d) *If $a \neq 0$ there is a connected neighborhood of ∞, N, such that $N \cap \sigma(\tau_a)$ consists of two unbroken simple arcs going to ∞; $N \cap S(a, +)$ and $N \cap S(a, -)$ are connected and unbounded.*

(e) *$S(a, +)$ is connected and unbounded. Every connected component of $S(a, -)$ contains a connected component of $\sigma(\tau_0)$ which in turn contains a nonempty subset of $\sigma(S(\theta, \tau_0))$ for every θ, $0 \leqslant \theta < 2\pi$. There are at most a finite number of connected components of $S(a, -)$.*

(f) *The set $\rho(\tau_0)$ is connected.*

PROOF. Statement (a) is a consequence of the remarks preceding Definition 4.2.

If $a \neq 0$ and $\lambda \in \sigma(\tau_a)$ then $v_1(\lambda) \neq v_2(\lambda)$ so both are analytic in a neighborhood of λ. Suppose our labeling is such that $| v_1(\lambda) | = e^{|a|}$. Since a nonconstant analytic function maps open sets onto open sets, every sufficiently small open neighborhood of λ is mapped by $v_1(\cdot)$ onto an open neighborhood of $v_1(\lambda)$. Thus every neighborhood of λ contains a point λ_1 such that $| v_1(\lambda_1) | > e^{|a|}$ and a point λ_2 such that $| v_1(\lambda_1) | < e^{|a|}$. Hence $\lambda_1 \in S(a, +)$, $\lambda_2 \in S(a, -)$ and (b) is proved.

Suppose U is a subset of $\rho(\tau_a)$. If both $U \cap S(a, +)$ and $U \cap S(a, -)$ are nonvoid then since $S(a, \pm)$ are open, U has a decomposition

$$U = [U \cap S(a, +)] \cup [U \cap S(a, -)]$$

into two nonvoid disjoint subsets, open in the relative topology of U, i.e., U is not connected. This proves (c).

To prove (d) let $0 \leqslant \theta < 2\pi$, let $\lambda_j(\theta) = (2\pi j + \theta)^2 + ia(2\pi j + \theta)$, $j = 0, \pm 1, \cdots$, let

$$d_j(\theta) = \inf_{l \neq j} | \lambda_j(\theta) - \lambda_l(\theta) |$$

and recall that in the course of proving Theorem 3.6(b) it was shown that there is a connected neighborhood of ∞, N, such that for K large enough, $\sigma(S(\theta, \tau_a)) \cap N = \{u_j(\theta)\}$, $| j | \geqslant K$ where $| u_j(\theta) - \lambda_j(\theta) | < d_j(\theta)/2$ and $\dim e(\theta, u_j(\theta), \tau_a) L_2(0, 1) = 1$, $| j | \geqslant K$. As θ increases from 0 to 2π, $\lambda_j(\theta)$ moves from $\lambda_j(0)$ to $\lambda_{j\pm 1}(0)$ and hence $u_j(\theta)$ must move continuously from $u_j(0)$ to $u_{j\pm 1}(0)$ so that $N \cap \sigma(\tau_a)$ consists of two connected components, corresponding to $j > K$ and $j < -K$ respectively. It follows from Theorem 4.4 that $\sigma(\tau_a) \cap N$ is locally homeomorphic to $(-1, 1)$ and thus we conclude that $\sigma(\tau_a) \cap N$ consists of two disjoint simple arcs, each approaching ∞. Hence $\rho(\tau_a) \cap N$ is made up of two connected unbounded components each of which from (c) lies wholly in $S(a, +)$ or $S(a, -)$. It follows from (b) that $S(a, +) \cap N$ and $S(a, -) \cap N$ are not void and hence the two connected components of $\rho(\tau_a) \cap N$ are just $S(a, +) \cap N$ and $S(a, -) \cap N$. Thus (d) is proved.

To prove the first assertion of (e) for $a \neq 0$ suppose $S(a, +)$ is not connected. Since, by (d), $S(a, +)$ has at most one unbounded connected component, this implies that $S(a, +)$ has a bounded connected component which we label B. But $| v_1(\lambda) | > e^{|a|}$, $| v_2(\lambda) | < e^{-|a|}$, $\lambda \in B$ for a suitable labeling of the characteristic values. Hence $v_1(\lambda) \neq v_2(\lambda)$, $\lambda \in B$ and thus $v_1(\lambda)$ is analytic in B. But this means we have an analytic function $v_1(\lambda)$ on a bounded domain B such that $\lim_{\lambda \to \bar{B} - B} | v_1(\lambda) | = e^{|a|}$ and $| v_1(\lambda) | > e^{|a|}$, $\lambda \in B$. Since $\bar{B}$ is compact, there exists a $\lambda_0 \in \bar{B}$ such that $| v_1(\lambda_0) | = \sup_{\lambda \in \bar{B}} | v_1(\lambda) |$ and clearly $\lambda_0 \in B$ a contradiction of the maximum modulus principle. Thus $S(a, +)$ is connected, $a \neq 0$. Now $0 \leqslant a_1 < a_2$ implies $S(a_1, +) \supseteq S(a_2, +)$ and furthermore

$$\bigcup_{a > 0} S(a, +) = \lim_{a \downarrow 0} S(a, +) = S(0, +).$$

Since $S(a, +)$ is connected and unbounded for $a \neq 0$, this implies that $S(0, +)$ is connected and unbounded. Thus the first assertion of (e) is proved.

If $a = 0$, $S(a, -)$ is void so the second assertion of (e) need be proved only for $a \neq 0$ which we assume. There is just one unbounded connected component of $S(a, -)$ by (b) and since $\sigma(\tau_0)$ is both unbounded and contained in $S(a, -)$, this unbounded connected component of $S(a, -)$ must contain elements of $\sigma(\tau_0)$. Suppose B is a bounded connected component of $S(a, -)$ and suppose $B \cap \sigma(\tau_0) = \phi$. Then for $\lambda \in \bar{B}$, $v_1(\lambda) \neq v_2(\lambda)$ and both are well defined analytic functions of λ, $\lambda \in \bar{B}$. Since $S(a, +)$ is connected, any connected component of $S(a, -)$ is simply connected and thus $\bar{B} - B$ is connected. Since $v_1(\lambda)$ and $v_2(\lambda)$ are continuous on $\bar{B}$ and $| v_i(\lambda) | = e^{-|a|}$ for $\lambda \in \bar{B} - B$, $| v_i(\lambda) | = \text{const.}$, $\lambda \in \bar{B} - B$, $i = 1, 2$. Suppose our labeling is such that $| v_1(\lambda) | = e^{-|a|}$, $\lambda \in \bar{B} - B$. Then by definition of $S(a, -)$, $| v_1(\lambda) | > e^{-|a|}$, $\lambda \in B$, a contradiction of the maximum modulus theorem. Thus $B \cap \sigma(\tau_0) \neq \phi$.

For $\lambda \in \sigma(\tau_0)$ let $C(\lambda)$ be the connected component of $\sigma(\tau_0)$ containing λ. Let C be any connected component of $S(a, -)$. Since $\bar{C} - C \in \sigma(\tau_a)$, $(\bar{C} - C) \cap \sigma(\tau_0) = \phi$ and thus (Lemma 4.5) if $\lambda \in C \cap \sigma(\tau_0)$ then $C(\lambda) \in C$, i.e., C contains a connected component of $\sigma(\tau_0)$. From the fact that the points in $\sigma(S(\theta, \tau_0))$ vary continuously as θ varies it is clear that any connected component of $\sigma(\tau_0)$ will contain an element of $\sigma(S(\theta, \tau_0))$, $0 \leqslant \theta < 2\pi$. Thus $C \cap \sigma(S(\theta, \tau_0)) \neq \phi$, $0 \leqslant \theta < 2\pi$, and the second assertion of (e) is proved.

Statement (d) implies that the bounded connected components of $S(a, -)$ are all contained in a compact set, say C. Since $C \cap \sigma(S(\theta, \tau_0)$ is finite, and each connected component of $S(a, -)$ contains a point in $\sigma(S(\theta, \tau_0)$, the number of connected components of $S(a, -)$ is finite, so (e) is proved.

The definition of $S(0, -)$ is inconsistent so $S(0, -) = \phi$. Thus $\rho(\tau_0) = S(0, +)$ which, by (e), is connected and unbounded. This proves (f). Q.E.D.

THEOREM 4.6. *The operator $T(\tau_a)$ is spectral for every real $a \neq 0$ if and only if $\sigma(\tau_0)$ has no bounded connected component.*

PROOF. Suppose that $T(\tau_a)$ is not spectral for some $a \neq 0$ and hence, since it is spectral at ∞, Theorem 4.4 tells us that there must exist a λ in $\sigma(\tau_a)$ in the neighborhood of which $\sigma(\tau_a)$ is not homeomorphic to $(-1, 1)$. In fact, letting our labeling be such that $| v_1(\lambda) | = e^a$, if $k \geqslant 1$ is the order of the zero of v_1' at λ then near λ, $\sigma(\tau_a)$ consists of $2(k + 1)$ rays emanating from λ. Thus from Theorem 4.5(b) we conclude that there is an $\epsilon > 0$ such that in the circle $N = \{z \mid | \lambda - z | < \epsilon\}$, the sets $\sigma(\tau_a)$, $S(a, +)$ and $S(a, -)$ have the configuration of Fig. 2 (where the two starred lines would be coincident if $k = 1$). Now $S(a, +)$ is connected hence given λ_1 and λ_2, as pictured in Fig. 2, we may construct a simple arc Γ_0 connecting them and lying wholly within $S(a, +)$. We may close this arc Γ_0 to a simple closed curve Γ by running arcs interior to $N \cap S(a, +)$, from λ_1 and λ_2 to λ. Since Γ' consists of one bounded and one unbounded connected component it is evident that one of the two pictured connected components of

$S(a, -) \cap N$ must lie in the bounded connected component of Γ'. It follows that $S(a, -)$ must have a bounded connected component B. This must, by Theorem 4.12(e), contain a bounded connected component of $\sigma(\tau_0)$.

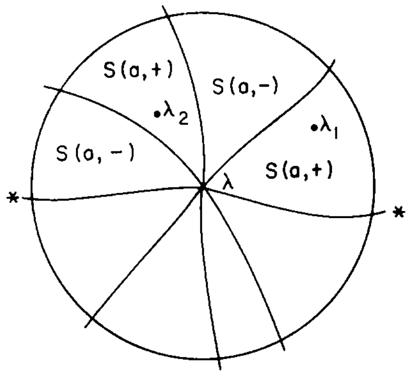

FIG. 2. The configuration of $\sigma(\tau_a)$, $S(a, +)$, and $S(a, -)$ near a branch point.

Suppose now that $\sigma(\tau_0)$ contains a bounded connected component σ_1. Let D be an open set containing σ_1 and such that $D \cap S(\tau_0) = \sigma_1$. Let $\Gamma = \bar{D} - D$. Since $\Gamma \cap \sigma(\tau_0) = \phi$, $w(\lambda) > 1$, $\lambda \in \Gamma$. Since Γ is compact and $w(\lambda)$ continuous, $\inf_{\lambda \in \Gamma} w(\lambda) > 1$. Let $\gamma > 0$ be such that $e_\gamma = \inf_{\lambda \in \Gamma} w(\lambda)$. Then for $0 < a < \gamma$, $S(a, -) \cap D \cap \sigma_1 \neq \phi$. But $S(a, -) \cap \Gamma = \phi$ hence $S(a, -)$ has a connected component contained in D. Thus, using Theorem 4.12(d), we know $S(a, -)$ has at least two connected components, $0 < a < \gamma$.

For large enough a, Theorem 3.6(a) and Lemma 4.1 assure us that (using the definitions of Theorem 4.5

$$(S(\theta, \tau_a)) - \{u_j(\theta)\}_{j=-\infty}^{\infty}, \qquad | u_j(\theta) - \lambda_j(\theta) | < \frac{d_j(\theta)}{2}$$

and

$$\dim e(\theta, u_j(\theta), \tau_a) L_2(0, 1) = 1, \qquad 0 \leqslant \theta < 2\pi.$$

As θ increases from 0 to 2π, the points $u_j(\theta)$ must move from $u_j(0)$ to $u_{j+1}(0)$ or $u_{j-1}(0)$ according as $j \geqslant 0$ or $j < 0$. Hence $\sigma(\tau_a)$ must consist of a single simple arc for large enough $| a |$. Thus by Theorem 4.5(b), c, for $| a |$ large enough, $S(a, -)$ is connected. If $0 < a_1 \leqslant a_2$ then $S(a_1, -) \subseteq S(a_2, -)$. Since every connected component of $S(a_2, -)$ contains points in $\sigma(\tau_0)$ and $\sigma(\tau_0) \subseteq S(a_1, -)$, every connected component of $S(a_2, -)$ contains at least one element of $S(a_1, -)$. Since the boundary of a connected component of $S(a_2, -)$ has void intersection with $S(a_1, -)$, this and Lemma 4.5 imply that every connected component of $S(a_2, -)$ must contain a connected component of $S(a_1, -)$. Hence $n(a)$ is a nonincreasing function of a, $0 < a < \infty$. Let $a_0 = \inf\{a \mid a > 0, n(a) = 1\}$. We conclude from the two preceding paragraphs that $0 < a_0 < \infty$. We will show that $T(\tau_{a_0})$ is not spectral on compact sets.

To do this we first establish that $n(a_0) > 1$, i.e., $S(a_0, -)$ is not connected. Since $n(a)$ is a nonincreasing step function for $a > 0$ which is finite for $a > 0$ and always jumps at least one when it jumps at all there is a b, $0 < b < a_0$ such that $n(a)$ is a constant k, $k > 1$, for $b \leqslant a < a_0$. Let $C_i(b)$, $1 \leqslant i \leqslant k$, be the connected components of $S(b, -)$. For $b \leqslant a < a_0$, each connected component of $S(a, -)$ has nonvoid intersection with $S(b, -)$ by Theorem 4.5(e). Since the boundary of $S(a, -)$ has void intersection with $S(b, -)$ it follows from Lemma 4.5 that each connected component of $S(a, -)$ contains a connected component of $S(b, -)$. Thus for $b \leqslant a < a_0$ there is a unique labeling of the connected components of $S(a, -)$ as $C_i(a)$, $1 \leqslant i \leqslant k$ where $C_i(a) \supseteq C_i(b)$, $1 \leqslant i \leqslant k$. Furthermore the same argument applied to a_1 instead of b where $b \leqslant a_1 \leqslant a_2 < a_0$ shows for such an a_1 and a_2 that $C_i(a_1) \subseteq C_i(a_2)$, $1 \leqslant i \leqslant k$. We have

$$S(a_0, -) = \bigcup_{b \leqslant a < a_0} S(a, -) = \bigcup_{b \leqslant a < a_0} \bigcup_{1 \leqslant i \leqslant k} C_i(a) = \bigcup_{1 \leqslant i \leqslant k} \bigcup_{b \leqslant a < a_0} C_i(a).$$

Let

$$C_i(a_0) = \bigcup_{b \leqslant a < a_0} C_i(a).$$

Since $S(a, -)$ is open, $C_i(a)$ is open, $a < a_0$ and thus $C_i(a_0)$ is open, $1 \leqslant i \leqslant k$. Suppose $\lambda \in C_i(a_0) \cap C_j(a_0)$, $i \neq j$. Then since $\{C_i(a)\}_a$ are growing, there exists in a, $b \leqslant a < a_0$ such that $\lambda \in C_i(a) \cap C_j(a)$ contradicting the disjointness of these sets. Thus $C_i(a_0) \cap C_j(a_0) = \phi$, $i \neq j$ and we conclude that $S(a_0, -)$ has the $k > 1$ connected components $C_i(a_0)$, $1 \leqslant i \leqslant k$ and hence is not connected.

Let U be the unbounded connected component of $S(a_0, -)$ and let $\Gamma = \bar{U} - U$. Then $\Gamma \subseteq \sigma(\tau_{a_0})$. Let d be the distance from Γ to $S(a_0, -) - U$. Suppose $d > 0$. Then there is a bounded open set E containing all the bounded connected components of $S(a_0, -)$ with $U \subseteq \bar{E}'$. Using an argument similar to that applied to the open set D above, we can conclude that there is an $a > a_0$ such that E contains a connected component of $S(a, -)$, contradicting the definition of a_0. Hence $d = 0$. Since $S(a_0, -) - U$ is bounded there is a $\lambda \in \Gamma$ such that $\text{dist}\{\lambda, S(a_0, -) - U\} = 0$. Thus for arbitrarily small circles N with centers λ, $S(a_0, -) \cap N$ is not connected. But this means that in no neighborhood of λ is $\sigma(\tau_{a_0})$ homeomorphic to $(-1, 1)$ and hence, by Theorem 4.4, $T(\tau_{a_0})$ is not spectral on compact sets. Q.E.D.

Let m, $0 \leqslant m \leqslant \infty$ be the number of bounded connected components of $\sigma(\tau_0)$. Since for $a > 0$ every connected component of $S(a, -)$ contains a connected component of $\sigma(\tau_0)$ and the unbounded connected component of $S(a, -)$ will contain all unbounded connected components of $\sigma(\tau_0)$, we have $n(a) \leqslant m - 1$, $a > 0$. We have also shown in the above proof that given any connected component σ_1 of $\sigma(\tau_0)$ there is a $\gamma > 0$ such that for $0 < a < \gamma$, $S(a, \ )$ has a connected component containing σ_1 and no other connected component of $\sigma(\tau_0)$. Hence given any finite number k of bounded connected components of $\sigma(\tau_0)$ there is an $\epsilon > 0$ such that for $0 < a < \epsilon$, $n(a) \geqslant k + 1$. It follows that $\lim_{a\to 0} n(a) = m - 1$. It is possible to show, as we have in a particular case above, that $T(\tau_a)$ is spectral, $a > 0$ if and only if the step function $n(\cdot)$ is continuous at a. Since $n(a)$ is monotone and finite for $a > 0$ and is an integer valued step function, given any $\epsilon > 0$ there are at most a finite number of points $a > \epsilon$ at which $n(\cdot)$ is not continuous. If $m = \infty$ then since $n(a) < \infty$ for $a > 0$ and $\lim_{a\to 0} n(a) = \infty$, it follows that $n(\cdot)$ must have a countably infinite number of points of discontinuity with 0 as their only accumulation point. We thus have the following theorem.

THEOREM 4.7. *Let m be the number of bounded connected components of $\sigma(\tau_0)$. There are at most m values of $a > 0$ for which $T(\tau_a)$ is not spectral. If $m = \infty$ the set of $a > 0$ for which $T(\tau_a)$ is not spectral is countably infinite and has 0 as its only accumulation point.*

REFERENCES

1. D. C. McGARVEY, Operators commuting with translation by one—Representation theorems. *J. Math. Anal. Appl.* 4 (1962), 366-410.
2. D. C. McGARVEY, Operators commuting with translation by one—Part II. Differential operators with periodic coefficients in $L_p(-\infty, \infty)$. *J. Math. Anal. Appl.*
3. J. SCHWARTZ, Perturbations of spectral operators, and applications I. Bounded perturbations. *Pacific J. Math.* 4 (1954), 415-458.
4. H. P. KRAMER, Perturbation of differential operators. *Pacfic J. Math.* 7 (1957), 1405-1435.
5. N. DUNFORD, Spectral operators. *Pacific J. Math.* 4 (1954), 321-354.
6. D. C. McGARVEY, Linear differential systems with periodic coefficients involving a large parameter, in press.
7. E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations." McGraw-Hill, New York, 1955.